Value

1. a) State the Density Theorem.

[1] $\forall x,y \in \mathbb{R}$ SUCH THAT $x<y$, $\exists r \in \mathbb{Q}$ WITH $x<r<y$

[1] b) State the Archimedean property.

$\forall x \in \mathbb{R}$, $\exists n_x \in \mathbb{N}$ SUCH THAT $x<n_x$

[2] c) State the definition of a limit of a sequence (x_n).

$\lim\limits_{n\to\infty} x_n = x$ IFF $\forall \varepsilon>0$, $\exists N \in \mathbb{N}$ S.T $\forall n \geq N$ WE HAVE THAT

$$|x_n - x| < \varepsilon$$

[5] d) Use a), b) and c) to show that $\forall x \in \mathbb{R}$, $\exists (r_n)$ depending on x, with $r_n \in \mathbb{Q}, \forall n \in \mathbb{N}$ and such that $\lim\limits_{n\to\infty} r_n = x$

LET $x \in \mathbb{R}$ SINCE $\frac{1}{n}>0$, $-\frac{1}{n}<0$ AND SO $x-\frac{1}{n}<x$, $\forall n \in \mathbb{N}$

BY DENSITY THM: $\exists r_n \in \mathbb{Q}$ WITH $x-\frac{1}{n}<r_n<x$ (OR. $x<r_n<x+\frac{1}{n}$)

<u>CLAIM:</u> $\lim\limits_{n\to\infty} r_n = x$

<u>PROOF</u>. LET $\varepsilon>0$ WE NEED TO FIND $N \in \mathbb{N}$ S.T $\forall n \geq N$, $|r_n - x|<\varepsilon$,

I.E $x-\varepsilon<r_n<x+\varepsilon$ BY b) (ARCHIMEDEAN PROPERTY) FOR $\frac{1}{\varepsilon}$, $\exists N_\varepsilon \in \mathbb{N}$

S.T $\frac{1}{\varepsilon}<N_\varepsilon$, I.E $\frac{1}{N_\varepsilon}<\varepsilon$ HENCE, $\forall n \geq N_\varepsilon$, $\frac{1}{n} \leq \frac{1}{N_\varepsilon}<\varepsilon$,

AND $-\varepsilon<-\frac{1}{N_\varepsilon} \leq -\frac{1}{n}$ BUT THEN $\forall n \geq N_\varepsilon$:

$x-\varepsilon<x-\frac{1}{N_\varepsilon} \leq x-\frac{1}{n}<r_n<x<x+\varepsilon$ (SINCE $\varepsilon>0$) I.E $\forall n \geq N_\varepsilon$

$x-\varepsilon<r_n<x+\varepsilon$, I.E $|r_n - x|<\varepsilon$ THUS, $\lim\limits_{n\to\infty} r_n = x$

2. Let $x_n = \dfrac{2n-1}{3n}, n \in \mathbb{N}$

[4] a) Show that (x_n) converges by ~~stating and~~ **using the definition** of limit of a sequence. AND THE ARCHIMEDEAN PROPERTY.

CLAIM: $\lim\limits_{n\to\infty} x_n = \frac{2}{3}$

PROOF: LET $\varepsilon > 0$. WE NEED TO FIND $N \in \mathbb{N}$ S.T $\forall n \geq N$, WE HAVE $|x_n - \frac{2}{3}| < \varepsilon$

BUT $|x_n - \frac{2}{3}| = |\frac{2n-1}{3n} - \frac{2}{3}| = |\frac{2}{3} - \frac{1}{3n} - \frac{2}{3}| = \frac{1}{3n}$, AND $\frac{1}{3n} < \varepsilon$ WHENEVER $n > \frac{1}{3\varepsilon}$ BY THE ARCHIMED. PROPERTY, $\exists N_\varepsilon \in \mathbb{N}$ S.T $\frac{1}{3\varepsilon} < N_\varepsilon$ THEN $\forall n \geq N_\varepsilon$, $n > \frac{1}{3\varepsilon}$ AND SO $|x_n - \frac{2}{3}| = \frac{1}{3n} < \varepsilon$

[5] b) Show that (x_n) is Cauchy by **stating and using the definition** of a Cauchy sequence. AND THE ARCHIMEDEAN PROPERTY

(x_n) IS A CAUCHY SEQUENCE IFF $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ S.T. $\forall n,m \geq N$, $|x_n - x_m| < \varepsilon$

$|x_n\ x_m| = |\frac{2n-1}{3n} - \frac{2m-1}{3m}| = |\frac{2}{3} - \frac{1}{3n} - \frac{2}{3} + \frac{1}{3m}| = \frac{1}{3}|\frac{1}{m} - \frac{1}{n}|$

WLOG: ASSUME $m > n$ I.E $\frac{1}{m} < \frac{1}{n}$ THEN $|\frac{1}{m} - \frac{1}{n} = \frac{1}{n}\ \frac{1}{m} < \frac{1}{n}$

BUT THEN $|x_n - x_m| = \frac{1}{3}\ \frac{1}{m} - \frac{1}{n}| < \frac{1}{3n}$ AND SO IF $\frac{1}{3n} < \varepsilon$ WE ARE DONE SAME AS ABOVE TAKE $N_\varepsilon \in \mathbb{N}$ SUCH THAT $\frac{1}{3\varepsilon} < N_\varepsilon$

THEN $\forall n, m \geq N_\varepsilon$, WITH $m > n$, WE HAVE THAT $|x_n - x_m| < \frac{1}{3n} < \varepsilon$. AND SO (x_n) IS CAUCHY

[3] c) Let (x_n) be a Cauchy sequence such that x_n is an integer, for all n in $\mathbb{N}$. Show that (x_n) is ultimately constant, i.e. $\exists N \in \mathbb{N}$ such that $x_n = x_N, \forall n \geq N$ (You can use the fact that $\forall k,s \in \mathbb{Z}$ such that $k \neq s$, $|k-s| \geq 1$.)

LET $x_n \in \mathbb{Z}$, $\forall n \in \mathbb{N}$ AND LET (x_n) BE CAUCHY LET $\varepsilon_0 = 1$ THEN $\exists N_{\varepsilon_0} \in \mathbb{N}$ S.T $\forall n,m \geq N_{\varepsilon_0}$, $x_n - x_m| < 1$ USING THE HINT (WITH $x_n = k$ $x_m = s$) IT MUST BE THAT $x_n = x_m$, I.E. $\forall n \geq N_{\varepsilon_0}$, $x_n = x_{N_{\varepsilon_0}}$

3. a) State the Bolzano-Weierstrass theorem.

[1]

EVERY BOUNDED SEQUENCE HAS A CONVERGENT SUBSEQUENCE

b) State the Nested Intervals Theorem.

[2] LET I_n BE A BOUNDED CLOSED INTERVAL AND $I_{n+1} \subseteq I_n$, $\forall n \in \mathbb{N}$.
THEN $\bigcap_{n=1}^{\infty} I_n \neq \phi$

[6] c) Show that the Bolzano-Weierstrass theorem implies the Nested Intervals Theorem. (Hint: choose $x_n \in I_n, \forall n \in \mathbb{N}$.)

LET THE BOLZANO WEIERSTRASS THEOREM BE TRUE, AND LET I_n BE CLOSED AND BOUNDED INTERVALS SUCH THAT $I_{n+1} \subseteq I_n$ $\forall n \in \mathbb{N}$. WE HAVE TO SHOW THAT $\bigcap_{n=1}^{\infty} I_n \neq \phi$

SINCE EACH $I_n \neq \phi$, LET $x_n \in I_n$, $\forall n \in \mathbb{N}$. ALSO, $x_n \in I_n \subseteq I_1 = [a_1, b_1]$ $\forall n \in \mathbb{N}$, AND SO $a_1 \leq x_n \leq b_1$ $\forall n \in \mathbb{N}$ THUS THE SEQUENCE (x_n) IS BOUNDED.

BY THE BOLZANO-WEIERSTRASS THM., LET (x_{n_k}) BE A CONVERGENT SUBSEQUENCE OF (x_n), I.E $\exists x \in \mathbb{R}$ S.T. $\lim_{k \to \infty} x_{n_k} = x$

WE WILL SHOW THAT THEN $x \in \bigcap_{n=1}^{\infty} I_n$, BY USING A PROOF BY CONTRADICTION.

SUPPOSE $x \notin \bigcap_{n=0}^{\infty} I_n$ THEN $\exists N \in \mathbb{N}$ SUCH THAT $x \notin I_N$, AND SINCE $I_n \subseteq I_N$, $\forall n \geq N$ $x \notin I_n$, $\forall n \geq N$. SINCE THE SET $\{n_k : k \in \mathbb{N}\}$ IS INFINITE AND STRICTLY INCREASING $\exists k_0 \in \mathbb{N}$ S.T. $\forall k \geq k_0$, $n_k \geq N$. NOTE THAT THEN $x_{n_k} \in I_{n_k} \subseteq I_N$, $\forall k \geq k_0$

SINCE $x \notin I_N = [a_N, b_N]$, EITHER $x < a_N$, OR $x > b_N$. IF $x < a_N$ I.E. $\varepsilon_1 = a_N - x > 0$, WE HAVE $x < a_N \leq x_{n_k} \leq b_N$, $\forall k \geq k_0$ AND SO $|x_{n_k} - x| = x_{n_k} - x \geq a_N - x = \varepsilon_1$, $\forall k \geq k_0$, WHICH CONTRADICTS $\lim_{k \to \infty} x_{n_k} = x$

IF $a_N \leq x_{n_k} \leq b_N < x$, $\forall k \geq k_0$, TAKE $\varepsilon_2 = x - b_N > 0$, AND SO $|x_{n_k} - x| =$
$= x - x_{n_k} \geq x - b_N = \varepsilon_2$, $\forall k \geq k_0$, WHICH AGAIN CONTRADICTS $\lim_{k \to \infty} x_{n_k} = x$

THUS, IT CAN NOT BE THAT $x \notin \bigcap_{n=1}^{\infty} I_n$, I.E. $x \in \bigcap_{n=1}^{\infty} I_n$

AND SO $\bigcap_{n=1}^{\infty} I_n \neq \phi$

4. a) Let A be a non-empty bounded above subset of $\mathbb{R}$ State the definition of $\sup A$ as given in class.

[2]

$s = \sup A$ IFF

1 $a \le s \ \forall a \in A$

2 IF $b < s$, $\exists a_b \in A$ SUCH THAT $b < a_b$.

[2] b) State the Squeeze Theorem for sequences.

LET $x_n \le y_n \le z_n$, $\forall n \in \mathbb{N}$ IF $\lim_{n\to\infty} x_n = L = \lim_{n\to\infty} z_n$, THEN $\lim_{n\to\infty} y_n = L$

[5] c) Let (x_n) be a bounded sequence, let $A = \{x_n : n \in \mathbb{N}\}$ and let $s = \sup A$ be such that s is not in A, i.e. $x_n < s, \forall n \in \mathbb{N}$ Use a) to show that if $b < s$ then the set $B_b = \{n : n \in \mathbb{N}, b < x_n < s\}$ must be infinite. (Hint: Suppose that B is finite and get a contradiction to $s = \sup A$ You can use the fact that every finite set has a maximal element.)

LET $b < s$ AND LET $B_b = \{n : n \in \mathbb{N}\ b < x_n < s\}$ BE A FINITE SET. NOTE THAT $B_b \neq \emptyset$ SINCE $s = \sup A$. LET $B_b = \{n_1, n_2, \dots, n_N\}$ FOR $N \in \mathbb{N}$. THEN THE (NONEMPTY) SET $A_b = \{x_{n_1}, \dots, x_{n_N}\}$ IS ALSO FINITE AND SO, LET $x_{n_i} = \max A_b$ NOW $\forall n \in \mathbb{N}$ EITHER $n \in B_b$, AND THEN $x_n \le x_{n_i} < s$ OR $n \notin B_b$, AND THEN $x_n \le b < x_{n_i} < s$

THUS x_{n_i} IS AN UPPER BOUND OF A AND $x_{n_i} < s$. THIS CONTRADICTS $s = \sup A$ I.E. THAT s IS THE SMALLEST UPPER BOUND OF A

HENCE, THE SET B_b MUST BE INFINITE

[5] d) Let (x_n), A and s be as in part c). Show that then there is a subsequence (x_{n_k}) of (x_n) that converges to s. (Hint: Use a) and c) to show that (IF $n_1 \in \mathbb{N}$ IS SUCH THAT $s-1 < x_{n_1} < s$) THEN $\forall k \ge 2, \exists x_{n_k} \in A$ such that $n_k > n_{k-1}$ and $s - \frac{1}{k} < x_{n_k} < s$ Then use the Squeeze Thm.)

SINCE $s = \sup A$ AND $s - 1 = b_1 < s$, $\exists n_1 \in \mathbb{N}$ WITH $s - 1 = b_1 < x_{n_1} < s$.

FOR $k=2$ AND $s - \frac{1}{2} = b_2 < s$, THE SET $B_{b_2} = \{n : n \in \mathbb{N}, s - \frac{1}{2} < x_n < s\}$ IS INFINITE AND SO WE CAN CHOOSE $n_2 \in B_{b_2}$ SUCH THAT $n_2 > n_1$, I.E $s - \frac{1}{2} < x_{n_2} < s$ AND $n_2 > n_1$

IF n_{k-1} IS SUCH THAT $s - \frac{1}{k-1} = b_{k-1} < x_{n_{k-1}} < s$, SINCE THE SET $B_{b_k} = \{n : n \in \mathbb{N}, s - \frac{1}{k} = b_k < x_n < s\}$ IS INFINITE $\exists n_k \in B_{b_k}$ SUCH THAT $n_k > n_{k-1}$ BUT THEN $s - \frac{1}{k} = b_k < x_{n_k} < s$ AND $n_k > n_{k-1}$ THUS (x_{n_k}) IS A SUBSEQUENCE OF (x_n) AND SINCE $s - \frac{1}{k} < x_{n_k} < s$, AND $\lim_{k\to\infty}(s - \frac{1}{k}) = s = \lim_{k\to\infty} s$, BY THE SQUEEZE THM: $\lim_{k\to\infty} x_{n_k} = s$