Value

1. Let $f\colon A \to B$ and let E be subset of A and let H be a subset of B.

a) State the definition of $f^{-1}(H)$ (inverse image of a set) and of $f(E)$

[2]

$f^{-1}(H) = \{x \in A : f(x) \in H\}$

$f(E) = \{f(x) \in B : x \in E\}$

b) State the definition of "f is an injection on A"

[1]

f IS AN INJECTION ON A IF $\forall x_1, x_2 \in A$ S.T $x_1 \neq x_2$, WE HAVE $f(x_1) \neq f(x_2)$

(OR IF $x_1, x_2 \in A$ SUCH THAT $f(x_1) = f(x_2)$, THEN $x_1 = x_2$)

c) Show that if f is an injection on A then $f^{-1}(f(E)) = E$ (Do not use the notion of "inverse function", only inverse image of a set. The proof should have two parts.)

[5]

PART 1: (SHOW THAT $E \subseteq f^{-1}(f(E))$ LET $e \in E$ (WE HAVE TO SHOW THAT $e \in f^{-1}(f(E))$.)

THEN $f(e) \in f(E) = H$ AND SO $e \in f^{-1}(H) = f^{-1}(f(E))$ THUS, $E \subseteq f^{-1}(f(E))$

PART 2: (SHOW THAT $f^{-1}(f(E)) \subseteq E$) LET $x \in f^{-1}(f(E))$. (WE HAVE TO SHOW THAT $x \in E$) THEN $f(x) \in f(E)$, i.e $\exists e \in E$ SUCH THAT $f(x) = f(e)$ SINCE f IS AN INJECTION (AND $x, e \in A$), WE HAVE THAT $x = e \in E$

THUS: $f^{-1}(f(E)) \subseteq E$

SINCE $E \subseteq f^{-1}(f(E))$ AND $f^{-1}(f(E)) \subseteq E$ WE HAVE THAT

$$f^{-1}(f(E)) = E$$

d) Give an example of f, A, B, E such that equality need not hold if f is not an injection.

[1]

LET $f(x) = x^2$, $A = B = \mathbb{R}$, $E = [0,2]$

THEN $f(E) = [0,4]$ AND $f^{-1}(f(E)) = [-2,2] \neq [0,2] = E$

2. Let $f : A \to B$

a) Define f is a surjection onto B.

[1]

$\forall b \in B, \exists a \in A$ SUCH THAT $b = f(a)$

b) If f is a bijection from A onto B, define the inverse function f^{-1}

[2] THE INVERSE FUNCTION f^{-1} IS SUCH THAT $D_{f^{-1}} = B$, $C_{f^{-1}} (= R_{f^{-1}}) = A$

AND $f^{-1}(y) = x \iff y = f(x)$, $\forall y \in B$

c) Prove that when f is a bijection, then f^{-1} is also a bijection. (You need to prove (i) f^{-1} is a surjection, and (ii) f^{-1} is an injection.)

[4]

(i) (PROOF OF f^{-1} IS A SURJECTION FROM B ONTO A)

LET $a \in A$ THEN $f(a) = b \in B \iff f^{-1}(b) = a$ AND SO

FOR $b = f(a) \in B$, WE HAVE THAT $a = f^{-1}(b)$

(ii) (PROOF OF f^{-1} IS AN INJECTION ON B)

LET $b_1, b_2 \in B$ BE SUCH THAT $f^{-1}(b_1) = f^{-1}(b_2) = a \in A$

BUT THEN $b_1 = f(a)$ AND $b_2 = f(a)$, AND SO $b_1 = f(a) = b_2$

d) Prove that when f is a bijection from A onto B, then $(f^{-1})^{-1}$ is the function f.

[3]

$D_f = A$, $C_f = R_f = B$, $D_{f^{-1}} = B$, $C_{f^{-1}} = R_{f^{-1}} = A$,

BUT THEN $D_{(f^{-1})^{-1}} = A = D_f$; $C_{(f^{-1})^{-1}} = R_{(f^{-1})^{-1}} = B = C_f = R_f$

ALSO, FOR $x \in A = D_{(f^{-1})^{-1}}$, $(f^{-1})^{-1}(x) = y \in B \iff f^{-1}(y) = x \iff y = f(x)$,

AND SO $\forall x \in A : (f^{-1})^{-1}(x) = f(x)$

3. In this question you are allowed to use only the field and order axioms and the mentioned theorems. **Clearly label which axiom or theorem you are using at each step.** Remember, addition and multiplication are binary operations, you need to use parenthesis to emphasize this fact.

Let a, b and c belong to an ordered field $\mathbb{R}$

a) Prove that $(a+(-b))+(b+(-c))=a+(-c)$

[4] $(a+(-b))+(b+(-c)) = ((a+(-b))+b)+(-c) = (a+((-b)+b))+(-c)$

$= (a+0)+(-c) = a+(-c).$

b) Show that if $a>b$ and $b>c$ then $a>c$ (Use the order axioms and a). Recall: $x>y$ means that $x-y\in\mathbb{P}$.)

[4] LET $a>b$ AND $b>c$, i.E $a\ b\in\mathbb{P}$ AND $b-c\in\mathbb{P}$ BUT THEN

By O1: $(a-b)+(b-c)\in\mathbb{P}$, i.E $a+(-b))+(b+(-c))\in\mathbb{P}$

By a): $(a+(-b))+(b+(-c))=a+(-c$ AND SO $a+(-c)=a-c\in\mathbb{P}$

THUS: $a>c$.

c) Prove that if $a>0$, then a^{-1} exists and $a^{-1}>0$ (Use the field and order axioms and the theorems stating that $1>0, \forall x,y,z\in\mathbb{R}, x\cdot 0=0$ and if $x>y$ and $z<0$, then $z\cdot x<z\cdot y$.)

[5] LET $a>0$ THEN BY O3: $a\neq 0$, AND SO BY M4, a^{-1} EXISTS.

IF $a^{-1}>0$, WE ARE DONE. IF NOT, THEN BY O3, EITHER $a^{-1}=0$, OR $a^{-1}<0$.

CLAIM 1: $a^{-1}\neq 0$, SINCE IF NOT, I.E IF $a^{-1}=0$ THEN $a\cdot a^{-1}=a\cdot 0 \overset{\text{THM.}}{=} 0$. BUT $a\cdot a^{-1}=1$, AND $1\neq 0$.

CLAIM 2: $a^{-1}<0$ CAN NOT BE TRUE, SINCE IF $a^{-1}<0$, USING THAT $a>0$, BY THE GIVEN THM. ABOVE: $a^{-1}\cdot a<a^{-1}\cdot 0$ BUT $a^{-1}\cdot a=1$ BY M4, $a^{-1}\cdot 0=0$ BY THE THM. ABOVE, AND SO WE WOULD HAVE $1<0$ (WHICH IS A CONTRADICTION TO $1>0$, BY O3.) SO $a^{-1}<0$ IS NOT TRUE.

THUS: $a^{-1}>0$

d) For $\varepsilon>0$, let $I_\varepsilon=[2-\varepsilon,2]=\{x\in\mathbb{R}: 2-\varepsilon\le x\le 2\}$ Show that $\bigcap_{\varepsilon>0} I_\varepsilon=\{2\}$

(Use the order axioms and the theorem which says that if $a-\varepsilon\le b, \forall\varepsilon>0$ then $a\le b$.)

[4] RECALL $x\in\bigcap_{\varepsilon>0} I_\varepsilon \Leftrightarrow x\in I_\varepsilon, \forall\varepsilon>0 \Leftrightarrow 2-\varepsilon\le x\le 2, \forall\varepsilon>0.$

FROM THE GIVEN THEOREM (WITH $a=2$ AND $b=x$), WE HAVE THAT THEN

$2\le x\le 2$, WHICH BY O3 GIVES THAT $x=2$

SO $x\in\bigcap_{\varepsilon>0} I_\varepsilon \Leftrightarrow x=2$, i.E $\bigcap_{\varepsilon>0} I_\varepsilon=\{2\}$

4. a) State the Principal of Mathematical Induction.
[2]

FOR A STATEMENT $P(n)$ SUCH THAT:

1. $P(1)$ is TRUE AND
2. $P(k)$ TRUE IMPLIES THAT THEN $P(k+1)$ is ALSO TRUE

WE HAVE THAT $P(n)$ is TRUE, $\forall n \in \mathbb{N}$.

b) State the Well Ordering Property.
[2]

IF $A \subseteq \mathbb{N}$, $A \neq \emptyset$, THEN $\exists m \in A$ SUCH THAT $m \leq k$, $\forall k \in A$

c) Prove that the Well Ordering Property implies the Principal of Mathematical Induction.
[6]

LET THE WOP BE TRUE AND LET $P(n)$ BE A STATEMENT SUCH THAT: 1) $P(1$ is TRUE AND
2) $P(k)$ TRUE IMPLIES THAT $P(k+1)$ is TRUE

WE HAVE TO SHOW THAT THEN $P(n)$ is TRUE $\forall n \in \mathbb{N}$, i.E THAT THE SET $S = \{n \in \mathbb{N} : P(n) \text{ is TRUE}\} = \mathbb{N}$

LET $A = \mathbb{N} \setminus S$ WE WILL SHOW THAT $A = \emptyset$.

SUPPOSE NOT, i.E SUPPOSE $A \neq \emptyset$. SINCE $A \subseteq \mathbb{N}$ THEN BY THE WOP: $\exists m \in A$ S.T $m \leq k$, $\forall k \in A$

SINCE $1 \in S$ WE HAVE $1 \notin A$ AND SO $m > 1$ BUT THEN $m-1 > 0$ i E $m-1 \in \mathbb{N}$ AND $m-1 < m$ SINCE $-1 < 0$ THUS $m-1 \notin A$, i E. $m-1 \in S$ BY P.M.I PART 2., THEN $P(m)$ is ALSO TRUE, i E $m \in S$ CONTRADICTION TO $m \in A = \mathbb{N} \setminus S$, i E $m \notin S$.

THUS $A = \emptyset$ AND SO $S = \mathbb{N}$