MATH 2080 Assignment No. 1, October 16, 2015

The assignment is due Friday, October 23, 2014, in class. Late assignments receive a mark of zero. Show all of your work and use the definitions given in class.

1. a) Prove that a supremum of a bounded above nonempty set A is unique, i.e. prove that if $s_1=\sup A$ and $s_2=\sup A$, then $s_1=s_2$. [1]
   b) Prove that every nonempty bounded below subset of $\mathbb{R}$ has an infimum. [3]
   c) Let A be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that a lower bound l of A is the infimum of A if and only if $\forall \varepsilon > 0, \exists a_\varepsilon \in A$ such that $a_\varepsilon < l + \varepsilon$. [3]

2. a) Show that if $\emptyset \neq A \subseteq B$ and B is bounded below, then $\inf B \leq \inf A$. [3]
   b) Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \bigcup B$ is a bounded set. Show also that then $\sup(A \bigcup B) = \sup\{\sup A, \sup B\}$. [4]
   c) Prove that for a nonempty, bounded above subsets A and B of $\mathbb{R}$, the set $A + B$ is also bounded above and that $\sup(A+B) = \sup A + \sup B$. [4]

3. Let $S = \left\{ \frac{1}{10^n} : n \in \mathbb{N} \right\}$. Use PMI to show that $\frac{1}{10^n} < \frac{1}{n}, \forall n \in \mathbb{N}$, and then use this, the definition of infimum and the corollary to the Archimedean property to show that $\inf S = 0$. (You cannot use limits of sequences.) [5]

4. Find the set A (proofs, **without using limits** are required):
   a) $A = \bigcap_{n=1}^{\infty} I_n$, for $I_n = (1, 2 + \frac{1}{n}]$, [4]
   b) $A = \bigcap_{n=1}^{\infty} I_n$, for $I_n = (2, 2 + \frac{1}{n})$, [3]
   c) $A = \bigcap_{n=1}^{\infty} I_n$, for $I_n = [n, \infty)$. [3]

5. a) Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Show that A is a nonempty bounded subset of $\mathbb{R}$. Find inf A and sup A. (Use the Corollary to the Archimedean property. You **cannot use limits**!) [3]

b) Let $A = \left\{ \frac{1}{n} - \frac{1}{m} ; n, m \in \mathbb{N} \right\}$. Show that A is a nonempty bounded set. Find inf A and sup A. (You can use part a), Q2. c), and a theorem proven in class on *inf* and *sup* of a product of a number with a set. You **cannot use limits**!) [3]

6. Let $\{I_k\}$ be a sequence of closed and bounded intervals, such that for any finite subset F of $\mathbb{N}$ we have $\bigcap_{k \in F} I_k \neq \varnothing$. Show that $\bigcap_{k=1}^{\infty} I_k \neq \varnothing$, by showing each of the following steps:

a) In general, if *I* and *J* are any two closed bounded intervals and $I \cap J \neq \varnothing$, then $I \cap J$ is a closed bounded interval. [4]

b) Use the Principal of Mathematical Induction to show that if $J_n = \bigcap_{k=1}^{n} I_k$, then J_n is a closed and bounded interval, for all $n \in \mathbb{N}$. [4]

c) Use the Nested Intervals Theorem to show that $\bigcap_{k=1}^{\infty} I_k \neq \varnothing$. [3]

Total [50 marks]