MATH 2202, Assignment No. 3
November 18, 2013

The assignment is due Monday, November 25, in class. Late assignments receive a mark zero.

1. a) Using the **definition of the limit** of a sequence prove that $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$. [4]
   b) Prove that if $\lim_{n\to\infty} x_n = 0$ and the sequence (y_n) is bounded, then $\lim_{n\to\infty} x_n y_n = 0$. [4]
   c) Give examples of sequences (x_n) and (y_n) such that $\lim_{n\to\infty} x_n = x \neq 0$, (y_n) is bounded, but $\lim_{n\to\infty} x_n y_n$ does not exist. [2]

2. Determine if (x_n) converges or diverges, find its accumulation points and $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$, if they exist. (**Explain** by showing all of your work and using theorems done in class. **Do not** use theorems on limits not done in class, which you might know from other courses, such as for example "L'Hospital's rule".)
   a) $x_n = \frac{\sin 2n}{n} + \frac{n^2}{1-n^2}$. [5]
   b) $x_n = (-1)^n \sqrt{n}(\sqrt{n+1} - \sqrt{n})$. [5]
   c) $x_n = \frac{2^n}{n}$. [5]

3. a) Show that $\lim_{x\to -1} \frac{x-5}{2x+3} = -6$, by using the definition of a limit. [5]
   b) Show that $\lim_{x\to 3} \frac{1}{3-x}$ does not exist, by showing that $f(x)=\frac{1}{3-x}$ is not bounded in any delta neighbourhood of 3. [5]
   c) Define g: $\mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$ for x rational and $g(x) = x - 4$ for x irrational. Find all points c at which $\lim_{x\to c} g(x)$ exists. Prove all of your statements. [5]

4. a) Let f: $\mathbb{R} \to \mathbb{R}$, let $c \in \mathbb{R}$, and let $\lim_{x\to c} f(x) = L > 0$. Show that there exists a neighbourhood $V_\delta(c)$ of c and $0< M<L$ such that for any x in $V_\delta(c) \setminus \{c\}$ we have that $f(x) > M$. [4]
   b) Let f: $A \to \mathbb{R}$ be nonnegative on A, and let c be a cluster point of A. Prove by using the definition of limit that if $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} \sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)}$. [5]

5. Let $f, g : A \to \mathbb{R}$ and let c be a cluster point of A.
   a) Show that if both $\lim_{x\to c} f(x)$ and $\lim_{x\to c}(f+g)(x)$ exist, then $\lim_{x\to c} g(x)$ exists. [4]
   b) If $\lim_{x\to c} f(x)$ and $\lim_{x\to c}(fg)(x)$ exist, does it follow that then $\lim_{x\to c} g(x)$ exists? Explain. [3]

Total [56]