Value

1. Let $f: A \to B$ and let G and H be subsets of B.

a) State the definition of $f^{-1}(G)$.

[1]

b) Prove that if $G \subseteq H$, then $f^{-1}(G) \subseteq f^{-1}(H)$.

[3]

c) Give an example of f, domain A, codomain B, and sets G and H such that $G \subseteq H$, $f^{-1}(G) = f^{-1}(H)$, but $G \neq H$.

[2]

2. a) Prove that for every a and b in $\mathbb{R}$, $-(a + b) = (-a) + (-b)$. State which field axioms or theorems you are using in each step.

[3]

b) Prove that if $a > 0$, then $a^{-1} > 0$. State the order and field axioms, or the theorems that you are using in each step.

[5]

3. a) Define when is a function f: A $\rightarrow$B an injection and when is it a surjection.

[2]

b) Prove in details that the set A = { $3n^2 + \sqrt{2}$: n in $\mathbb{N}$ } is denumerable (i.e. infinite countable). (You have to find a bijection f between $\mathbb{N}$ and A and show that that f is a bijection.)

[5]

4. a) Prove, by using the Principal of Mathematical Induction, that $\frac{1}{2^n} < \frac{1}{n}$, for every n in $\mathbb{N}$.

[4]

b) Define the infimum of a set A that is bounded from below.

[2]

c) Show that the infimum of the set A = $\{\frac{1}{2^n} : n \in \mathbb{N}\}$ is 0. (Hint: use a corollary of the Archimedean property and part a).)

[4]

d) Find the set A = $\bigcap_{n=1}^{\infty} I_n$, with $I_n = [-1, \frac{1}{2^n}]$, by using c)

[2] and the conclusion in the proof of the Nested Intervals Theorem, as done in class.

5. a) State the definition of $\lim_{n\to\infty} x_n = x$.

[2]

b) State the definition of a bounded sequence. State the Theorem on convergence and boundedness of sequences.

[2]

c) Prove that the sequence with $x_n = n$ diverges. (Hint: use proof by contradiction, part b) and the Archimedean Theorem. State the Archimedean Theorem.) [4]