Value

1. a) Let $f: A \to B$ be an injection and let E and F be subsets of A. Give the definition of "f is an injection", and show that then $f(E) \cap f(F) \subseteq f(E \cap F)$. Show that this might not be true if f is not an injection by providing an example of
[5] f, A, B, E and F with $f(E) \cap f(F) \not\subseteq f(E \cap F)$.

LET $x \in f(E) \cap f(F)$. THEN $x \in f(E)$ AND $f \in f(F)$, i.e. $\exists a \in E$, $\exists b \in F$ SUCH THAT $x = f(a) = f(b)$. SINCE f IS AN INJECTION, IT MUST BE THAT $a = b$, AND SO $a \in E \cap F$. BUT THEN $x = f(a) \in f(E \cap F)$ AND SO $f(E) \cap f(F) \subseteq f(E \cap F)$.

EXAMPLE: $f(x) = x^2$, $A = B = \mathbb{R}$, $E = [-2, -1]$, $F = [1, 2]$.
THEN $E \cap F = \emptyset$, $f(E \cap F) = \emptyset$, $f(E) = [1, 4]$, $f(F) = [1, 4]$
AND SO $f(E) \cap f(F) = [1, 4] \not\subseteq f(E \cap F) = \emptyset$.

b) Using only the field axioms of $\mathbb{R}$, show that for $a, b \in \mathbb{R}$, if $a + b = 0$, then $b = -a$, and that furthermore this implies $(-1)a = -a$. (You can also use the THM: $0a = 0$)

[5]

LET $a, b \in \mathbb{R}$ AND $a + b = 0 \;/ + (-a)$ $\Leftrightarrow -a + (a + b) = -a + 0$,
(USING A2, AND A3) $\Leftrightarrow (-a + a) + b = -a$ (USING A4)
$\Leftrightarrow 0 + b = -a$ (USING A3) $\Leftrightarrow b = -a$.

NOW $a + (-1)a \overset{M3}{=} 1 \cdot a + (-1)a \overset{D}{=} (1 + (-1))a = 0 \cdot a \overset{THM}{=} 0$.

FROM WHAT WE HAVE PROVEN ABOVE; FOR $b = (-1)a$, WE GET THAT $(-1)a = -a$.

2. a) State the Archimedean property and give the definition of $\inf A$ for a non-empty, bounded below subset A of $\mathbb{R}$. [3]

b) Let $A = \{x \in \mathbb{R} : x > 2\}$. Show that A is non-empty, bounded below and not bounded above subset of $\mathbb{R}$. Show that $\inf A = 2$ by using the definition of infimum and a corollary to the Archimedean property, which you should state explicitly. [5]

SINCE $2<3$ (FOLLOWS FROM $0<1$ AND THM'S ON ORDER), $3 \in A$ AND SO $A \neq \phi$; $2<x$, $\forall x \in A$, AND SO 2 is A LOWER BOUND OF A;

A is NOT BOUNDED ABOVE SINCE FOR ANY $M \in \mathbb{R}$, BY THE ARCH. PROPERTY, $\exists N_1 \in \mathbb{N}$ SUCH THAT $N_1 > 2$ AND $\exists N_2 \in \mathbb{N}$ S.T. $N_2 > M$. THEN, IF $N = \max\{N_1, N_2\}$, WE HAVE THAT $N > 2$, AND SO $N \in A$, AND ALSO $N > M$.

THUS, NO $M \in \mathbb{R}$ CAN BE AN UPPER BOUND OF A.

PROOF OF $\inf A = 2$: (i) $2 < x$, $\forall x \in A$ AND SO 2 is A LOWER BOUND.

(ii) IF $2 < u$, THEN $u - 2 > 0$, AND BY A COROLL. TO THE ARCH. PROPERTY, $\exists n \in \mathbb{N}$ SUCH THAT $0 < \frac{1}{n} < u-2$. BUT THEN:

$2 < \frac{1}{n} + 2 < u$, $\frac{1}{n} + 2 \in A$, AND SO u is NOT A LOWER BOUND OF A.

THUS 2 is THE SMALLEST UPPER BOUND OF A, i.e. $2 = \inf A$.

c) Find a bijection between A and $\mathbb{R}$ to support the claim that A is uncountable. (You can use functions that you have seen in calculus. No proof is required here.) [2]

A is UNCOUNTABLE, SINCE $f(x) = e^x + 2$ is A BIJECTION FROM $\mathbb{R}$ ONTO A. THUS: $|A| = |\mathbb{R}|$.

3. a) For a sequence of real numbers (x_n), give the definition of $\lim_{n\to\infty} x_n = x$.

[2]

[6] b) Show that $\forall x \in \mathbb{R}$, $\exists (r_n^x)$ with $r_n^x \in \mathbb{Q}, \forall n \in \mathbb{N}$ and such that $\lim_{n\to\infty} r_n^x = x$.

LET $x \in \mathbb{R}$. FOR EACH $n \in \mathbb{N}$, $x < x + \frac{1}{n}$ (SINCE $n > 0 \Rightarrow \frac{1}{n} > 0$).

BY THE DENSITY THEOREM, $\forall n \in \mathbb{N}$, $\exists r_n^x \in \mathbb{Q}$ SUCH THAT $x < r_n^x < x + \frac{1}{n}$. CLAIM: $\lim_{n\to\infty} r_n^x = x$.

PROOF: LET $\varepsilon > 0$. BY COROLL. TO THE ARCH. PROPERTY, $\exists N \in \mathbb{N}$ SUCH THAT $0 < \frac{1}{N} < \varepsilon$. THEN $\forall n \geq N$, WE HAVE THAT $\frac{1}{n} \leq \frac{1}{N}$ AND SO $\underline{x - \varepsilon} < x < r_n^x < x + \frac{1}{n} < \underline{x + \varepsilon}$, i.e. $|r_n^x - x| < \varepsilon$.

THUS, THIS $N = N(\varepsilon)$ WOULD DO.

4. Let $x_n = \dfrac{n-1}{2n}, n \in \mathbb{N}$.

[5] a) Show that (x_n) converges by using the definition of limit of a sequence.

WE HAVE TO SHOW THAT $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ S.T. $\forall n \geq N$ WE HAVE THAT $|x_n - \frac{1}{2}| < \varepsilon$. (i.e. THAT $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$.)

NOTE: $|x_n - \frac{1}{2}| = \left|\frac{n-1}{2n} - \frac{1}{2}\right| = \left|\frac{n-1-n}{2n}\right| = \frac{1}{2n} < \varepsilon$, WHENEVER $n > \frac{1}{2\varepsilon}$. THUS, TAKING $N = [\frac{1}{2\varepsilon}] + 1$, WE HAVE THAT $\forall n \geq N$, $|x_n - \frac{1}{2}| = \frac{1}{2n} \leq \frac{1}{2N} < \varepsilon$, AND SO THIS $N = N(\varepsilon)$ WOULD DO.

[5] b) Show that (x_n) is Cauchy by stating and using the definition of a Cauchy sequence.

WE HAVE TO SHOW THAT $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ SUCH THAT $\forall n, m \geq N$ WE HAVE THAT $|x_n - x_m| < \varepsilon$.

ASSUME WLOG: $m > n$

NOTE: $|x_n - x_m| = \left|\frac{n-1}{2n} - \frac{m-1}{2m}\right| = \left|\cancel{\frac{1}{2}} - \frac{1}{2n} - \cancel{\frac{1}{2}} + \frac{1}{2m}\right| = \frac{1}{2n} - \frac{1}{2m} <$
$< \frac{1}{2n} < \varepsilon$, WHENEVER $n > \frac{1}{2\varepsilon}$. THUS, TAKING $N = [\frac{1}{2\varepsilon}] + 1$, WE HAVE THAT WHENEVER $m > n \geq N$, $|x_n - x_m| < \frac{1}{2N} < \varepsilon$, AND SO, THIS $N = N(\varepsilon)$ WOULD DO. (IF $n > m$; $|x_n - x_m| = \frac{1}{2m} - \frac{1}{2n} < \frac{1}{2m} \leq \frac{1}{2N} < \varepsilon$ WHENEVER $n > m \geq N$, i.E. THE SAME N WOULD DO.)

5. a) State the Bolzano-Weierstrass theorem.

[2]

b) Use a) and the definition of a limit to show that if a bounded sequence of real numbers (x_n) diverges, and if (x_{n_k}) is a subsequence of (x_n) such that $\lim_{k\to\infty} x_{n_k} = a$, then there must exist another subsequence (x_{n_s}) of (x_n), and a real number $b \neq a$ such that $\lim_{s\to\infty} x_{n_s} = b$.

[8] LET (x_n) DIVERGE, AND LET (x_{n_k}) BE A SUBSEQUENCE OF (x_n) SUCH THAT $\lim_{k\to\infty} x_{n_k} = a$.

SINCE (x_n) DIVERGES, $\exists\, \varepsilon_0 > 0$, SUCH THAT $\forall N \in \mathbb{N}$, $\exists n \geq N$ WITH $|x_n - a| \geq \varepsilon_0$. WE WILL CREATE A SUBSEQUENCE OF (x_n) WHICH IS OUTSIDE OF $V_{\varepsilon_0}(a)$. FOR $N=1$, LET $n_1 \geq 1$, SUCH THAT $|x_{n_1} - a| \geq \varepsilon_0$. FOR $N = n_1 + 1$, LET $n_2 \geq n_1 + 1 > n_1$, SUCH THAT $|x_{n_2} - a| \geq \varepsilon_0$. .... FOR $N = n_s + 1$, LET $n_{s+1} \geq n_s + 1 > n_s$ BE SUCH THAT $|x_{n_{s+1}} - a| \geq \varepsilon_0$. ....

THUS (x_{n_s}) IS A SUBSEQUENCE OF (x_n), AND $x_{n_s} \notin V_{\varepsilon_0}(a)$, $\forall s \in \mathbb{N}$.

NOW (x_{n_s}) IS A BOUNDED SEQUENCE, SINCE (x_n) IS BOUNDED, AND BY BOLZ.-WEIERSTR., $\exists (x_{n_{s_l}})$, SUBSEQUENCE OF (x_{n_s}) (AND SO ALSO A SUBSEQ. OF (x_n)), WHICH CONVERGES TO SOME $b \in \mathbb{R}$.

NOTE THAT THEN $b \neq a$, SINCE FOR $\varepsilon = \varepsilon_0$, $\exists\, l_0 \in \mathbb{N}$ S.T. $\forall l \geq l_0$ WE HAVE THAT $x_{n_{s_l}} \in V_{\varepsilon_0}(b)$ (AND $x_{n_{s_l}} \notin V_{\varepsilon_0}(a)$, $\forall l \in \mathbb{N}$).