MATH 2202   Assignment No.2, October 16, 2013

The assignment is due Wednesday, October 23, 2013, in class. Late assignments receive mark zero.

1. Use the **Principle of Mathematical Induction** to prove that:
   a) $5^{2n}-1$ is divisible by 8, for every n in $\mathbb{N}$.   [4]
   b) $n+1<2^n<n!$, for every $n \geq 4$.   [4]
   c) Let $a>0, b>0$. Show that $a<b$ if and only if $\forall n \in \mathbb{N}$ we have that $a^n<b^n$. [3]

2. a) Using the field and the order axioms show that if $x<y$, then $x<(x+y)/2<y$. [5]
   b) If $a<x<b$ and $a<y<b$, show that $|x-y|<b-a$. Interpret this geometrically by referring to the distances between points in $\mathbb{R}$. [5]
   c) Determine and sketch the set of pairs (x,y) in $\mathbb{R}\times\mathbb{R}$ that satisfy $|x|=|y|$. [3]

3. a) Prove that for a nonempty, bounded below subsets A and B of $\mathbb{R}$,
   $inf\,(A+B) = inf\,A + inf\,B$.   [5]
   b) Prove that a supremum of a bounded nonempty set A is unique, i.e. prove that if $s_1=supA$ and $s_2=supA$, then $s_1=s_2$.   [3]

4. a) Prove that if A is a bounded non empty subset of $\mathbb{R}$, there exists a **smallest** closed interval I containing A. (In other words, I has the property that if J is a closed interval containing A, then J contains I.)   [5]
   b) Give an example of a non empty bounded set A, subset of $\mathbb{R}$, such that there is no smallest open interval containing A.   [2]

5. Find *supA* and *infA*, whenever they exist. In parts a) and b) **first find** A by using a corollary to the Archimedean theorem. (You can use theorems proven in class on *sup* of an interval, on *inf* and *sup* of: sum of sets; product of a number with a set. You **cannot use limits**!)
   a) $A=\bigcap_{n=1}^{\infty} I_n$ with $I_n=(1,2+\frac{1}{n}]$,   [5]
   b) $A=\bigcap_{n=1}^{\infty} I_n$ with $I_n=(1,1+\frac{1}{n})$,   [4]
   c) $A=\{\frac{1}{n}-\frac{1}{m}: n,m \text{ in } \mathbb{N}\}$. (Hint: find first *inf* and *sup* of $\{\frac{1}{n}: n \in \mathbb{N}\}$.) [5]

6. Let $\{I_k\}$ be a sequence of closed and bounded intervals such that for any finite set $F\subset\mathbb{N}$, we have $\bigcap_{k\in F} I_k \neq \varnothing$. Show that then $\bigcap_{k=1}^{\infty} I_k \neq \varnothing$, by showing each of the following steps:

a) In general, if I and J are any two closed bounded intervals and $I \bigcap J \neq \varnothing$, then $I \bigcap J$ is a closed bounded interval. [4]

b) Use the Principal of Mathematical Induction to show that if $J_n = \bigcap_{k=1}^{n} I_k$, then J_n is a closed and bounded interval, for all $n \in \mathbb{N}$. [4]

c) Use the Nested Intervals Theorem to show that $\bigcap_{k=1}^{\infty} I_k \neq \varnothing$. [3]

Total [64 marks]