1. Let f: A $\to$ B and let H be a subset of the co-domain B.

a) Give the definitions of $f^{-1}(H)$ and f a surjection.

b) Prove that if f is surjective, then $f(f^{-1}(H)) = H$.

c) Give an example of f , A, B and H such that $H \not\subset f(f^{-1}(H))$.

2. Let $F(+,\bullet)$ be an ordered field with zero $\hat{0}$ and identity $\hat{1}$.
Prove, by using the field and order axioms, that for x, y, z in F:

a) $x \bullet y=\hat{0}$ if and only if $x=\hat{0}$ or $y=\hat{0}$. (You can use that $z\bullet \hat{0}=\hat{0}$, for every z in F.)

b) Let x^2 be defined as $x \bullet x$. Prove that $x^2+y^2=\hat{0}$ if and only if $x=\hat{0}$ and $y=\hat{0}$.
(You can use part a) and the fact that if $x \neq \hat{0}$, then $x^2 \in \mathbb{P}$.)

3. Prove, by using the Principal of Mathematical Induction, that every non-empty finite subset of $\mathbb{R}$ has a greatest member (maximum). (Namely, if A = $\{a_1, a_2, \ldots, a_n\}$, then there exists a_i in A such that $a_i \geq a_j$, for j =1, 2, …n. You can use that if $a \geq b$ and $b \geq c$ then $a \geq c$.)

4. a) Define the infimum of a set A that is bounded from below.

b) State the Archimedean property.

c) Show, by using the **definition** of infimum and part b), that the infimum of the set A = $\left\{\frac{3-2x}{x} : x \geq 1\right\}$ is equal to -2.

5. a) State the definition of limit of a sequence.

b) Prove by using the definition that $\lim\limits_{n\to\infty} \dfrac{3n}{2n+1} = \dfrac{3}{2}$.

c) Prove that if $\lim\limits_{n\to\infty} x_n = 0$ and (y_n) is bounded, then $\lim\limits_{n\to\infty} x_n y_n = 0$.

6. a) State the Nested Intervals theorem.

b) Give an example of a sequence of nested, bounded, open intervals I_n such that $\bigcap\limits_{n=1}^{\infty} I_n = \varnothing$.

c) Give an example of a sequence of nested, unbounded, closed intervals I_n such that $\bigcap\limits_{n=1}^{\infty} I_n = \varnothing$. (Note: $[a, \infty)$ is a closed interval.)

d) State the Bolzano-Weierstrass theorem.

e) Give an example of an unbounded sequence that has a bounded subsequence.

f) Give an example of a bounded sequence that does not converge.

7. (a) State the definition of $\lim_{x \to c} f(x) = L$ (where $f : A \to \mathbb{R}$ and c is a cluster point of A.)

b) Prove by using the **definition** that $\lim_{x \to -1} \frac{x+2}{x-3} = \frac{-1}{4}$.

c) Let f: A$\to \mathbb{R}$ be nonnegative on A, and let f be continuous at c. Prove by using the **definition** of limit that then $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)}$,

8. Let f, g : $\mathbb{R} \to \mathbb{R}$ and let h = f+g and k = f–g. Use the Theorem on limits of combination of functions to prove that if h and k are continuous at c, then f and g are also continuous at c.

9. a) Define when is f uniformly continuous on A.

b) Prove by using the definition that $f(x) = \dfrac{1}{x^2}$ is uniformly continuous on A=$[1, \infty)$.

10. Let f be continuous on $(-2, 2]$ and such that $\lim_{x\to -2} f(x)$ does not exist. Answer the following questions and **explain your answer. State the Theorems, if you are using any.**

a) Does $\lim_{n\to\infty} f((2-\frac{1}{n})^2 - 3)$ exist? If yes, is it equal to $\lim_{n\to\infty} f\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)$?

b) If $f(-1) = -f(1)$ can the range of f equal to $(3, \infty)$?

c) Can the image of $[-1, 2]$ under f be equal to $[0, 3)$?

d) Is f uniformly continuous on $(-2, 2)$? Is it uniformly continuous on $[-1, 2]$?

e) If (x_n) is a Cauchy sequence in $[-1, 2]$, must $(f(x_n))$ also be Cauchy?

f) Give an example of f as above and (x_n) in $(-2, 2]$ such that (x_n) is a Cauchy sequence but $f(x_n)$ is not Cauchy.