MATH 2202, Assignment No. 3
November 14, 2014

The assignment is due Friday, November 21, in class. Late assignments receive a mark zero.

1. Let (x_n) and (y_n) be sequences of positive numbers and suppose that $\lim_{n\to\infty} (x_n/ y_n) = 0$.
   a) Show that if $\exists N \in \mathbb{N}, \exists c > 0$, such that $x_n \ge c, \forall n \ge N$, then $\lim_{n\to\infty} y_n = \infty$. [5]
   b) Use a) to show that if $\lim_{n\to\infty} x_n = \infty$, then $\lim_{n\to\infty} y_n = \infty$. [2]

2. a) Show that a is an accumulation point for (x_n) if and only if $\forall \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \ge N$ such that $|x_n - a| < \varepsilon$. [5]
   b) Let $\lim_{n\to\infty} x_n = L$. Show that if $\exists N \in \mathbb{N}, \forall n \ge N, x_n \ne L$, then L is a cluster point of $A = \{x_n : n \in \mathbb{N}\}$. Give an example of a convergent sequence (x_n) with its limit L a cluster point of A, but such that it is not true that $\exists N \in \mathbb{N}, \forall n \ge N, x_n \ne L$. [5]

3. Show that if (x_n) is a bounded sequence then $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = l$ if and only if the sequence (x_n) converges and $\lim_{n\to\infty} x_n = l$. [5]

4. a) Show that $\lim_{x\to -2} \dfrac{x-3}{6+2x} = \dfrac{-5}{2}$, by using the definition of a limit. [5]
   b) Show that $\lim_{x\to 5} \dfrac{2}{x-5}$ does not exist, by showing that $f(x)=\dfrac{2}{x-5}$ is not bounded in any delta neighbourhood of 5. [4]
   c) Define g: $\mathbb{R} \to \mathbb{R}$ by $g(x) = 3x$ for x rational and $g(x) = x + 2$ for x irrational. Find all points c at which $\lim_{x\to c} g(x)$ exists. Prove all of your statements. [4]

5. a) Let $f: A \to \mathbb{R}$ be nonnegative on A, and let c be a cluster point of A. Prove by using the definition of limit that if $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} \sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)}$. [5]
   b) Let f: $\mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and such that $f(m / 2^n) = 0$ for all m in $\mathbb{Z}$ and all n in $\mathbb{N}$. Show that f(x) = 0, for all x in $\mathbb{R}$. (Hint: show first that $\forall x > 0, \forall \varepsilon > 0$ there exist n, m in $\mathbb{N}$ such that $\left| x - \dfrac{m}{2^n} \right| < \varepsilon$.) [5]
   c) Let f: $\mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let S= {x in $\mathbb{R}$: f(x) = 0}. If (x_n) is contained in S and x = $\lim_{n\to\infty} x_n$, show that x is also in S. [3]

Total [48]