1. Let $X=[1,\infty)$ and let ρ be a function on $X\times X$ defined by $\rho(x,y)=\left|\frac{1}{x}-\frac{1}{y}\right|$.

3 a) Show that ρ is a metric on X.

$\rho: X\times X \to \mathbb{R}^+\cup\{0\}$ and is well defined. $\forall x,y,z\in X$:

(i) $\rho(x,y)=0 \iff \left|\frac{1}{x}-\frac{1}{y}\right|=0 \iff \frac{1}{x}=\frac{1}{y} \iff x=y$

(ii) $\rho(x,y)=\left|\frac{1}{x}-\frac{1}{y}\right| = \left|\frac{1}{y}-\frac{1}{x}\right| = \rho(y,x)$

(iii) $\rho(x,y)=\left|\frac{1}{x}-\frac{1}{y}\right| = \left|\frac{1}{x}-\frac{1}{z}+\frac{1}{z}-\frac{1}{y}\right| \le \left|\frac{1}{x}-\frac{1}{z}\right|+\left|\frac{1}{z}-\frac{1}{y}\right| = \rho(x,z)+\rho(z,y)$

So: ρ is a metric

1+3 b) Give the definition of a Cauchy sequence. Show that the sequence (x_n) with $x_n=n$, for $n\in\mathbb{N}$, is a Cauchy sequence in (X,ρ).

Let $\varepsilon>0$. Since $\rho(n,m)=\left|\frac{1}{n}-\frac{1}{m}\right|\le\frac{1}{n}+\frac{1}{m}$,
by Archemedean Thm: $\exists N_\varepsilon\in\mathbb{N}$ s.t. $\frac{1}{N_\varepsilon}<\frac{\varepsilon}{2}$. But then $\forall n,m\ge N_\varepsilon$, say WLOG, if $n\ge m$, we have $\rho(x_n,x_m)\le\frac{1}{n}+\frac{1}{m}\le\frac{2}{m}\le\frac{2}{N_\varepsilon}<\frac{2\varepsilon}{2}=\varepsilon$.

c) Show that (X,d_x) is complete, where $d_x=d/X\times X$ is the restriction of the usual metric (Euclidean) metric d on $\mathbb{R}$, (i.e. (X,d_x) is a subspace of $(\mathbb{R},d)$). (State and use a theorem from class. You can also use that $(\mathbb{R},d)$ is complete.)

3 Theorem: (Y,d) a metric space, $X\subseteq Y$. Then if (Y,d) is complete and X is a closed subset of Y, (X,d_x) is complete

$X=[1,\infty)$ is closed in $(\mathbb{R},d)$; $(\mathbb{R},d)$ is complete

$\Rightarrow (X,d_x)$ is complete.

2. Let $B = \{f : [0,1] \to \mathbb{R}; f \text{ is bounded}\}$ and let $d_\infty(f,g) = \sup\{|f(x) - g(x)|; x \in [0,1]\}$.

For $r \geq 1$, let $f_r(x) = \begin{cases} 2 - \frac{1}{r}, & x \in (\frac{1}{2}, 1] \\ 1, & x \in [0, \frac{1}{2}] \end{cases}$, and $S = \{f_r; r \in [1, \infty)\}$ be a subset of B.

5

a) Show that S is not closed in B. (You can use any equivalent definition of closed set, but you have to state which one are you using.) [2]

S is not closed since $\exists (f_n) \subseteq S$, $f_n \to f$, but $f \notin S$.

$f_n(x) = \begin{cases} 2 - \frac{1}{n}, & x \in (\frac{1}{2}, 1] \\ 1, & x \in [0, \frac{1}{2}] \end{cases}$. If $x \in [0, \frac{1}{2}]$: $f_n(x) = 1$, $\forall n$

If $x \in (\frac{1}{2}, 1]$, $f_n(x) = 2 - \frac{1}{n} \to 2$, $n \to \infty$. Thus $f(x) = \begin{cases} 2, & x \in (\frac{1}{2}, 1] \\ 1, & x \in [0, \frac{1}{2}] \end{cases}$

is such that $f_n \to f$ in (B, d_∞): $\forall \varepsilon > 0$, $\forall n \geq N_\varepsilon$, with $\frac{1}{N_\varepsilon} < \varepsilon$

we have $d_\infty(f_n, f) = \sup\{|f_n(x) - f(x)| : x \in [0,1]\} = \sup\{0, \frac{1}{n}\} =$

$= \frac{1}{n} \leq \frac{1}{N_\varepsilon} < \varepsilon$.

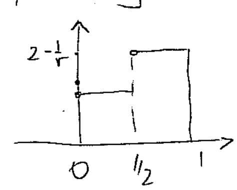
But $f \notin S$, since $\nexists\, r \in [1, \infty)$ such that $2 - \frac{1}{r} = 2$.

5

b) Show that S is not open in B.

S is not open in B. For any $r \in [1, \infty)$, and any $\varepsilon > 0$

define $h_\varepsilon(x) = \begin{cases} 2 - \frac{1}{r}, & x \in (\frac{1}{2}, 1] \\ 1, & x \in (0, \frac{1}{2}] \\ 1 + \varepsilon/2, & x = 0 \end{cases}$



Then $h_\varepsilon \in B \setminus S$ and $h_\varepsilon \in N_{d_\infty}(f_r; \varepsilon)$ since

$d_\infty(h_\varepsilon, f_r) = \sup\{|h_\varepsilon(x) - f_r(x)|, x \in [0,1]\} = \sup\{\varepsilon/2\} = \varepsilon/2 < \varepsilon$.

3. Let (X,d) be a metric space. For $S \subseteq X$ define boundary of S by:
x is in *bdr(S)* if $\forall r > 0$, $N(x; r) \cap S \neq \varnothing$ and $N(x; r) \cap S^C \neq \varnothing$,
where S^C denotes the complement of S in X.

5

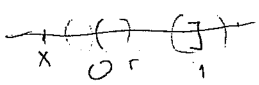
a) Show that S is open if and only if $S \cap bdr(S) = \varnothing$.

Let S be open. Then $\forall x \in S$, $\exists N_d(x;r_x) \subseteq S$, i.e. $N_d(x;r_x) \cap S^c = \phi$ and so $x \notin bdr(S)$. So $S \subseteq (bdr(S))^c$, i.e. $S \cap bdr(S) = \phi$.

Let $S \cap bdr(S) = \phi$. Now for any $x \in S$, $x \notin bdr(S)$ and so: $\exists r_x > 0$ such that either $N_d(x;r_x) \cap S = \phi$, or $N_d(x;r_x) \cap S^c = \phi$. But since $x \in N(x;r_x) \cap S$, it must be that $N(x;r_x) \cap S^c = \phi$, i.e. $N(x;r_x) \subseteq S$. So: S is open.

5

b) If $X = \mathbb{R}$ with the usual metric and $S = (0, 1]$, show that $bdr(S) = \{0, 1\}$. (Don't forget that you have to show that 0 and 1 are boundary points of S, and also that there are no other boundary points of S.)



First show $\{0,1\} \subseteq bdr(S)$: $\forall r > 0$, $N(0;r) \cap S \ni \frac{r}{2}$ and $-\frac{r}{2} \in N(0;r) \cap S^c$. Also: $\forall r > 0$, $1 \in N(1;r) \cap S$ and $1 + \frac{r}{2} \in N(1;r) \cap S^c$.

Now: $bdr(S) \subseteq \{0,1\}$: Let $x \neq 0$, $x \neq 1$. Then $r_0 = \min\{|x|, |x-1|\} > 0$ and if $x \notin S$: $N(x;r_0) \cap S = \phi$; if $x \in S$, then $N(x;r_0) \cap S^c = \phi$. In both cases: $x \notin bdr(S)$.

6

4. Show that if (X,ρ) and (Y,σ) are two discrete metric spaces, i.e. ρ and σ are the discrete metrics on X and on Y respectively, then the product metric d on $X\times Y$ is equivalent to the discrete metric on the product space. (You can use a theorem from class determining the open sets in the product space. State that theorem. Then recall which are the open sets in the topology induced by the discrete metric.)

Thm: U is open in $(X\times Y, d)$ iff U is a union of sets of type $U_X \times U_Y$, with U_X open in X; U_Y open in Y.

$\mathcal{T}(\rho)=\mathcal{P}(X)$; $\mathcal{T}(\sigma)=\mathcal{P}(Y)$ (Every subset of X (or Y) is open in X (or Y).)

But then every subset of $X\times Y$ is of the form $S = \bigcup_{(x,y)\in S} \{(x,y)\} = \bigcup_{(x,y)\in S} \{x\}\times\{y\}$ ($\{x\}$ open, $\{y\}$ open) $\Rightarrow$ S is open in $X\times Y$.

5. a) Give three equivalent definitions of a closed set in a metric space.

3

S is closed $\Leftrightarrow$ S^c is open

$\Leftrightarrow S' \subset S \Leftrightarrow (\forall (x_n)\subseteq S,\ x_n\to x \text{ in } X \Rightarrow x\in S)$

b) Prove that if (X,d_x) and (Y,d_y) are metric spaces and $f: X\to Y$ is (d_x,d_y) continuous, then for every closed set F in Y, we have that $f^{-1}(F)$ is closed subset of X.

6

Let $f: X\to Y$ be (d_x, d_y) continuous, let F be closed in Y.

We want to show that $f^{-1}(F)$ is closed subspace of X.

Proof 1: (Use seq. criteria): Let $(x_n)\subseteq f^{-1}(F)$ and let $x_n\to x$ in X.

Then $f(x_n)\in F$, f is contin. at $x \Rightarrow f(x_n)\to f(x)$ in Y and since F is closed, it must be that $f(x)\in F$.

But then $x\in f^{-1}(F)$ and we are done.

Or: Proof 2: (Or use: f contin. iff $f^{-1}(U)$ is open, $\forall U$ open in Y): Let F be closed in Y. We want $f^{-1}(F)$ is closed, i.e. $(f^{-1}(F))^c$ is open.

Since F^c is open in Y, $f^{-1}(F^c)$ is open in X. Now $f^{-1}(F^c) = (f^{-1}(F))^c$, since $x\in f^{-1}(F^c) \Leftrightarrow f(x)\in F^c \Leftrightarrow f(x)\notin F \Leftrightarrow x\notin f^{-1}(F) \Leftrightarrow x\in (f^{-1}(F))^c$; and so $(f^{-1}(F))^c$ is open, i.e. $f^{-1}(F)$ is closed.

6. Let $X = C_{[0,1]}$ be the metric space of all continuous functions on [0,1] with the d_∞ metric. max
Let ϕ be a fixed continuous function from X, and let $F_\phi : X \to X$ be defined by: for f in X

$$F_\phi(f)(x) = \phi(x) f(x), \ x \text{ in } [0,1].$$

a) State the ε, δ definition of continuity of a function between metric spaces at a point in the domain space.

2

f is continuous at $x_0 \in X$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$ such that whenever $d_X(x, x_0) < \delta$, we have $d_Y(f(x), f(x_0)) < \varepsilon$.

b) Show that the function F_ϕ as defined above, is continuous at every point of X. by using a)
(Hint: the fixed function ϕ is bounded on [0,1], and attains its maximum value $M \geq 0$. Look at the cases $M = 0$, and $M > 0$ separately.)

5

Case $M>0$: Let $f_0 \in X$, let $\varepsilon > 0$. Since $d_\infty(F(f), F(f_0)) =$
$= \max\{|\phi(x) f(x) - \phi(x) f_0(x)| : x \in [0,1]\} \leq M \max\{|f(x) - f_0(x)| : x \in [0,1]\}$
$= M\, d_\infty(f, f_0)$, if we take $\delta = \frac{\varepsilon}{M}$, we are done.

Case $M=0$: $\Rightarrow \phi(x) \equiv 0$ on $[0,1]$. But then for $\varepsilon > 0$, any δ would do, since $d_\infty(F(f), F(f_0)) = d_\infty(0,0) = 0 < \varepsilon$ (always).

c) Give the definition of a homemorphism between two metric spaces.
If ϕ is the function identically equal to 0 on [0,1], can F_ϕ from above be a homemorphism? Explain.

3

$f: X \to Y$ is homeomorphism if f is a bijection and both f and f^{-1} are continuous.

F_ϕ is not a homeomorphism, since F_ϕ is not a bijection (F_ϕ is not 1-1 : $F_\phi \mathbb{1} = 0 = F_\phi e_1$ for $e_1(z) = z$.)