1. Prove that if *(X, d)* is a metric space and $S \subseteq X$, then $d(x, S)=0$ if and only if $x \in Cl(S)$. [7]

$\Rightarrow$ Let $d(x,S)=\inf\{d(x,y): y \in S\}=0$. If $x \in S$, then we are done. If $x \notin S$, we have to show that $x \in S'$. Let $\varepsilon > 0$. Since $0 = \inf\{d(x,y): y \in S\}$, for $0 < \varepsilon$, $\exists y \in S$, such that $0 \leq d(x,y) < \varepsilon$, i.e. $\exists y \in N(x;\varepsilon) \cap S = N(x;\varepsilon) \cap (S \setminus \{x\})$. Thus $x \in S'$, i.e. $x \in cl(S)$.

$\Leftarrow$ Let $x \in cl(S) = S \cup S'$. If $x \in S$, then $d(x,x)=0$, and so $\inf\{d(x,y): y \in S\}=0$. If $x \in S' \setminus S$, then $\forall \varepsilon > 0$, $N(x;\varepsilon) \cap (S \setminus \{x\}) = N(x;\varepsilon) \cap S \neq \phi$, and so $\exists y \in S$ such that $0 \leq d(x,y) < \varepsilon$. Thus, no $\varepsilon > 0$ is a lower bound of $\{d(x,y): y \in S\}$, and since $0 \leq d(x,y)$, $\forall y \in S$; we have that 0 is the greatest lower bound, i.e. $0 = \inf\{d(x,y): y \in S\}$.

2. Let $(V, \|\ \|)$ be a nonzero normed vector space, let θ be the zero vector of V, and let

$$B(\theta;1) = \{v \in V: \ \|v - \theta\| = \|v\| \leq 1\}$$

be the closed unit ball of V. Also, for u and v in V, define $f_{uv}: [0,1] \to V$, by

$$f_{uv}(t) = (1-t)u + tv, \ \forall t \in [0,1].$$

a) Show that for fixed u and v in V, the function f_{uv} is a Lipshitz map from $[0,1]$ **into** $B(\theta;1)$ (which shows that f_{uv} is also continuous, i.e. that f_{uv} is a path from u to v in $B(\theta;1)$). [6]

Let $s, t \in [0,1]$. Then $d(f_{uv}(t), f_{uv}(s)) = \|f_{uv}(t) - f_{uv}(s)\| =$

$= \| tu + (1-t)v - su - (1-s)v\| = \|(t-s)u + \cancel{v} - \cancel{v} - tv + sv\| =$

$= \|(t-s)u - (t-s)v\| = |t-s| \, \|u-v\| \leq |t-s| (\|u\| + \|v\|) \leq |t-s| (1+1)$

$= 2|t-s|$. Thus f_{uv} is LIPSHITZ, WITH LIPSHITZ CONSTANT 2.

ALSO: $\forall t \in [0,1]$, $\| tu + (1-t)v - \theta \| \leq t\|u\| + (1-t)\|v\| + \|\theta\|$

$\leq t \cdot 1 + (1-t) \cdot 1 + 0 = 1$ (since $\|u\| \leq 1$, $\|v\| \leq 1$, $\|\theta\| = 0$)

i.e. $f_{uv}(t) \in B(\theta;1)$, $\forall t \in [0,1]$

b) Show that the closed unit ball $B(\theta;1)$ of $(V, \|\ \|)$ is path connected. (Hint: use a).) [2]

$B(\theta;1)$ IS PATH CONNECTED BY a): $\forall u, v \in B(\theta;1)$, f_{uv} FROM a) IS CONTINUOUS; $f_{uv}(0) = u$, $f_{uv}(1) = v$, AND $f_{uv}(t) \in B(\theta;1)$, $\forall t \in [0,1]$. (i.e. f_{uv} IS A PATH FROM u TO v, IN $B(\theta;1)$)

c) Is $B(\theta;1)$ connected? Explain your answer. [2]

YES: BY A THM: EVERY PATH CONNECTED SPACE IS CONNECTED.

3. a) Give the definition of a separable metric space and the definition of a Lindelof metric space (X, d). [3]

(X, d) IS SEPARABLE IF $\exists D \subseteq X$, D COUNTABLE AND DENSE IN X.
(X, d) IS LINDELOF IF EVERY OPEN COVER OF X HAS A COUNTABLE SUBCOVER

b) Let (X, d) be a Lindelof metric space. Show that then (X, d) is separable by following the next three steps:
(i) Show that $\forall n \in \mathbb{N}$, there exists a countable subset $D_n = \{y_{ni} : i = 1,2,3...\}$ of X, such that D_n is a $\frac{1}{n}$ - net for X, i.e. D_n is such that $X = \bigcup_{i=1}^{\infty} N(y_{ni}; \frac{1}{n})$.
(ii) Show that $D = \bigcup_{n=1}^{\infty} D_n$ is dense in X, by showing that $\forall \varepsilon > 0$, D is an ε-net for X.
(iii) Show that (X, d) is separable by using (i) and (ii). [7]

LET (X, d) BE LINDELOF.

(i) $\forall n \in \mathbb{N}$, $\frac{1}{n} > 0$ AND $X = \bigcup_{x \in X} N(x; \frac{1}{n})$, i.e. $\mathcal{N}_n = \{N(x; \frac{1}{n}) : x \in X\}$ IS AN OPEN COVER OF X. So: $\exists$ COUNTABLE SUBCOVER $\mathcal{N}_n' = \{N(x_{ni}; \frac{1}{n}) : x_{ni} \in X, i \in \mathbb{N}\}$
LET $D_n = \{x_{ni} : i \in \mathbb{N}\}$. NOTE: $X = \bigcup_{i=1}^{\infty} N(x_{ni}; \frac{1}{n})$.

(ii) LET $D = \bigcup_{n=1}^{\infty} D_n$. WE WILL SHOW THAT D IS DENSE IN X: LET $\varepsilon > 0$. LET $n_\varepsilon \in \mathbb{N}$ BE SUCH THAT $\frac{1}{n_\varepsilon} < \varepsilon$. FROM (i): $X = \bigcup_{i=1}^{\infty} N(x_{n_\varepsilon i}; \frac{1}{n_\varepsilon}) \subseteq$ $\subseteq \bigcup_{i=1}^{\infty} N(x_{n_\varepsilon i}; \varepsilon)$, with $x_{n_\varepsilon i} \in D_{n_\varepsilon} \subseteq D$, $\forall i \in \mathbb{N}$. So, D IS AN ε-NET OF X, $\forall \varepsilon > 0$, i.e. D IS DENSE IN X.

(iii) X IS SEPARABLE SINCE $D = \bigcup_{n=1}^{\infty} D_n$ IS BOTH DENSE IN X (BY (ii)) AND ALSO COUNTABLE, SINCE COUNTABLE UNION OF COUNTABLE SETS D_n IS COUNTABLE.

4. a) Give the definition of a totally bounded metric space (X, d). [2]

(X,d) IS TOTALLY BOUNDED IF $\forall \varepsilon>0, \exists F_\varepsilon \subseteq X$, F_ε FINITE AND SUCH THAT $X = \bigcup_{y \in F_\varepsilon} N(y;\varepsilon)$.

b) Show that if (X, d) is totally bounded and $f:(X,d) \to (Y,\rho)$ is uniformly continuous, then $f(X)$ is totally bounded subspace of the metric space (Y, ρ). [7]

LET f BE UNIFORMLY CONTINUOUS, AND LET $\varepsilon>0$. (WE WANT TO SHOW THAT $\exists G_\varepsilon \subseteq f(X)$, G_ε FINITE AND AN ε-NET FOR X.) FOR THE GIVEN $\varepsilon>0$, $\exists \delta>0$ SUCH THAT WHENEVER $d(x_1,x_2)<\delta \Rightarrow d(f(x_1),f(x_2))<\varepsilon$.

SINCE X IS TOTALLY BOUNDED, FOR $\delta>0$, $\exists F_\delta = \{x_1, x_2, \dots, x_n\}$ SUCH THAT $X = \bigcup_{i=1}^{n} N(x_i;\delta)$. BUT THEN, SINCE $f(N_d(x_i;\delta)) \subseteq N_\rho(f(x_i);\varepsilon)$ $\forall i \in \{1,2,\dots,n\}$, WE HAVE THAT $f(X) = f\left(\bigcup_{i=1}^{n} N_d(x_i;\delta)\right) = \bigcup_{i=1}^{n} f(N_d(x_i;\delta)) \subseteq$
$\subseteq \bigcup_{i=1}^{n} N_\rho(f(x_i);\varepsilon)$, I.E. $f(X) = \bigcup_{i=1}^{n} \left(N_\rho(f(x_i);\varepsilon) \cap f(X)\right)$, AND SO $G_\varepsilon = \{f(x_1), f(x_2), \dots, f(x_n)\}$ IS A FINITE ε-NET OF $f(X)$.

THUS, $f(X)$ IS TOTALLY BOUNDED.

5. Fill in the table below by writing YES or NO in each of the empty boxes:

☺	Connected	Path connec.	Separable	Bounded	Totally bdd.
$\mathbb{N}$	No	No	Yes	No	No
[0, 1]	Yes	Yes	Yes	Yes	Yes
$\mathbb{R}^n$	Yes	Yes	Yes	No	No
Closed unit ball of l^2	Yes	Yes	Yes	Yes	No
Closed unit ball of l^∞	Yes	Yes	No	Yes	No
Closed unit ball of C[0,1]	Yes	Yes	Yes	Yes	No

[6]