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[2] 1. (a) Define what is meant by a "linear equation in variables $x_1, ..., x_n$".

> **Solution:** A linear linear equation in variables $x_1, ..., x_n$ is an equation of the form $a_1x_1 + \cdots + a_nx_n = b$, where $a_1, \ldots, a_n, b$ are real numbers.

[2] (b) Define what it means for an $n \times n$ matrix A to be *invertible*.

> **Solution:** An $n \times n$ matrix A is invertible if and only if there exists an $n \times n$ matrix B so that $AB = I_n = BA$.

[2] (c) What is a system of linear equations of the form $A\mathbf{x} = \mathbf{0}$ called?

> **Solution:** Such a system is called a *homogeneous* system

[2] (d) Define what is meant by "A is symmetric". (Be brief!)

> **Solution:** A matrix A is symmetric if and only if $A = A^T$

[3] (e) Give an example of an inconsistent system of linear equations.

> **Solution:**
>
> one example is: $\begin{array}{rrl} x & +y & = 0 \\ 2x & +2y & = 2 \end{array}$
>
> There are other examples.

[2] 2. For a system of equations of the form $A\mathbf{x} = \mathbf{0}$, is it

✓ always consistent,

❏ sometimes consistent,

❏ or inconsistent?

Check the appropriate box and explain.

> **Solution:** A system $A\mathbf{x} = \mathbf{0}$ always has the trivial solution of $\mathbf{x} = \mathbf{0}$, so such a system is always consistent.

[4] 3. Let A be an $n \times n$ matrix. State four *additional* properties equivalent to "A is invertible".

> **Solution:** (See Theorem 1.5.3, 1.6.4, 2.3.3, or 2.3.6)
>
> - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
> - The RREF of A is I_n.
> - A is expressible as a product of elementary matrices.
> - For every $n \times 1$ matrix $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ is consistent.
> - For every $n \times 1$ matrix $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has a unique solution.
> - $\det(A) \neq 0$.

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[3] 4. Prove that if a matrix A has an inverse, then this inverse is unique. Give reasons for each step.

Solution: Let A be $n \times n$ be invertible with two possible inverses, say B and C. Because each is an inverse then (a) $AB = I_n$, (b)$BA = I_n$, (c)$AC = I_n$ (d) $CA = I_n$. Then

$$B = BI_n = B(AC) = (BA)C = I_n C = C$$

where the reasons for the successive equalities are: I_n is the multiplicative identity, (c), associativity of multiplication, (b), and again using the identity.

[2] 5. What is the maximum number of 0's in an invertible 5×5 matrix? Explain.

Solution: 20 is the maximum. This maximum is attained by the invertible matrix I_5. If there are more than 20 zeros, some row (or column) must contain all zeros, in which case the matrix is not invertible (since it will have a determinant of zero).

[7] 6. (a) By Gauss-Jordan elimination, solve the (consistent) system

$$\begin{array}{rcrcrl} x & + & 2y & - & z = & 3, \\ 4x & + & 10y & + & 2z = & 10, \\ x & + & 3y & + & 2z = & 2. \end{array}$$

If row operations are not clearly and properly identified, your answers will not be graded. If there is more than one solution, state the solution using parameter(s).

Solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 4 & 10 & 2 & 10 \\ 1 & 3 & 2 & 2 \end{array}\right) \Longrightarrow \begin{array}{lcl} R_2 & \to & R_2 - 4R_1 \\ R_3 & \to & R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 2 & 6 & -2 \\ 0 & 1 & 3 & -1 \end{array}\right) \Longrightarrow R_2 \to \tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 3 & -1 \end{array}\right) \Longrightarrow \begin{array}{lcl} R_1 & \to & R_1 - 2R_2 \\ R_3 & \to & R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -7 & 5 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So the solutions are $\begin{array}{rcl} x & = & 5 + 7t \\ y & = & -1 - 3t \\ z & = & t \end{array}$ where $t \in \mathbb{R}$.

[1] (b) Find a particular solution and verify that it is indeed a solution.

Solution: If we let $t = 0$ we get the solution $x = 5, y = -1, z = 0$. (There are others.) This is verified by

$$\begin{array}{rcrcrl} (5) & + & 2(-1) & - & (0) = & 3, \\ 4(5) & + & 10(-1) & + & 2(0) = & 10, \\ (5) & + & 3(-1) & + & 2(0) = & 2. \end{array}$$

[8] 7. Let $A = \begin{bmatrix} 4 & 0 & -1 \\ 1 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & -2 \\ 0 & 0 \\ -1 & 7 \end{bmatrix}$.

Calculate each of the following, and if the expression is not defined, say why.

[2] (a) The size of A^TC^T= $\underline{3 \times 3}$

[2] (b) AC.

Solution: $AC = \begin{bmatrix} 13 & -15 \\ 3 & -2 \end{bmatrix}$

[2] (c) B^{-1} (by any method).

Solution: $B^{-1} = \frac{1}{(4)(1)-(-3)(0)} \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ 0 & 1 \end{bmatrix}$

[2] (d) A^TC.

Solution: This is undefined. Since A is a 2×3 matrix, A^T is a 3×2 matrix; C is also a 3×2 matrix. The product is undefined because the number of columns of A^T is not the same as the number of rows of C.

[2] 8. For what values of a and b is the matrix $M = \begin{bmatrix} 1 & a \\ a & b \end{bmatrix}$ an elementary matrix?

Solution: $a = 0$, $b \neq 0$.

9. Calculate the following determinants by any method; only answers are graded:

[1] (a) $\begin{vmatrix} 1 & -5 \\ 7 & 6 \end{vmatrix} = \underline{41}$.

[2] (b) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \underline{-1}$.

[3] (c) $\begin{vmatrix} 1 & 0 & -5 & 2 \\ 0 & 6 & 1 & 0 \\ 3 & -6 & -2 & -3 \\ 1 & 0 & 0 & -1 \end{vmatrix} = \underline{18}$.

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[7] 10. Suppose that A is a 5×5 matrix and $\det(A) = -2$. Find (only answers are graded):

[2] (a) $\det(A^{-1}) = \underline{-\frac{1}{2}}$;

[1] (b) $\det(A^4) = \underline{16}$;

[2] (c) $\det(3A) = \underline{3^5(-2) = -486}$;

[2] (d) $\det(\text{adj}(A)) = \underline{16}$.

[7] 11. Let $H = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$. Given that $\det(H) = 12$, compute

[3] (a) $\text{cof}(H)$, the cofactor matrix for H,

Solution: $\text{cof}(H) = \begin{bmatrix} -6 & 3 & 2 \\ -6 & -3 & 6 \\ 6 & -3 & 2 \end{bmatrix}$

[2] (b) $\text{adj}(H)$, the adjoint of H, and

Solution: $\text{adj}(H) = \begin{bmatrix} -6 & -6 & 6 \\ 3 & -3 & -3 \\ 2 & 6 & 2 \end{bmatrix}$

[2] (c) H^{-1} (using the adjoint).

Solution: $H^{-1} = \frac{1}{12} \begin{bmatrix} -6 & -6 & 6 \\ 3 & -3 & -3 \\ 2 & 6 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$