Mathematics MATH1300
Vectors Geometry and Linear Algebra
Midterm Examination
October 20, 2008, 5:30–6:30pm

1. (20%)

   (a) Solve by **Gauss-Jordan** Elimination:

$$\begin{aligned} 2x - 4y - 6z &= 2 \\ 3x + 5y + 2z &= -8 \end{aligned}$$

Answer: Take $\begin{bmatrix} 2 & -4 & -6 & 2 \\ 3 & 5 & 2 & -8 \end{bmatrix}$,

- multiply row 1 by $\frac{1}{2}$ ($R_1 \leftarrow \frac{1}{2}R_1$) to get $\begin{bmatrix} 1 & -2 & -3 & 1 \\ 3 & 5 & 2 & -8 \end{bmatrix}$
- Subtract three times row 1 from row 2 ($R_2 \leftarrow R_2 - 3R_1$) to get $\begin{bmatrix} 1 & -2 & -3 & 1 \\ 0 & 11 & 11 & -11 \end{bmatrix}$
- Multiply row 2 by $\frac{1}{11}$ ($R_2 \leftarrow \frac{1}{11}R_2$) to get $\begin{bmatrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$
- Add twice row 2 to row 1 ($R_1 \leftarrow R_1 + 2R_2$) to get $\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Hence z is a free variable, say $z = t$. It then follows that $x = -1+t$ and $y = -1 - t$.

   (b) Solve the following system of linear equations **using Cramer's rule**:

$$\begin{aligned} 4x_1 - 11x_2 &= 5 \\ 2x_1 - 5x_2 &= 2 \end{aligned}$$

Answer: $\det \begin{bmatrix} 4 & -11 \\ 2 & -5 \end{bmatrix} = 2$, $\det \begin{bmatrix} 5 & -11 \\ 2 & -5 \end{bmatrix} = -3$ and $\det \begin{bmatrix} 4 & 5 \\ 2 & 2 \end{bmatrix} = -2$. Hence $x_1 = -\frac{3}{2}$ and $x_2 = -1$.

2. (15%) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$. If the given expression is defined, calculate the resulting matrix or value. Otherwise, give a reason why it does not exist.

(a) AB

Answer: undefined since the matrices have the wrong shape

(b) CA^T

Answer: $\begin{bmatrix} -2 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$

(c) $B(A+C)$

Answer: undefined since $A+C$ undefined

3. (15%) Let $B = \begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix}$.

(a) Suppose A satisfies the equation $2A^T + I = B$. Find A.

Answer: $2A^T = \begin{bmatrix} 0 & 4 \\ -2 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$

(b) Find B^{-1}

Answer: $B^{-1} = \frac{1}{13} \begin{bmatrix} 5 & -4 \\ 2 & 1 \end{bmatrix}$

(c) Find adj(B).

Answer: $\text{adj}(B) = \begin{bmatrix} 5 & -4 \\ 2 & 1 \end{bmatrix}$

4. (20%) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$.

(a) Find A^{-1}

Answer: $A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

(b) Consider the elementary row operation where row 2 is replaced by row 2 minus twice row 1 (sometimes written $R_2 \leftarrow R_2 - 2R_1$). Let C be the matrix we get after applying this operation to A Find the elementary matrix E such that $C = EA$

Answer: $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) The determinant of A is -2. Find the determinant of $B = 3A^TA^3$ using the properties of determinants (don't multiply the matrices to find B; you may leave your answer as a product of integers).

Answer: $\det(B) = \det(3A^TA^3) = \det(3A^T)\det(A^3) = 3^3 \det(A^T)\det(A)^3 = 3^3 \cdot (-2)^4 = 27 \cdot 16.$

5. (15%)

(a) Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ -2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

Answer: Expanding on the third column gives

$$\det(A) = -3 \det\left(\begin{bmatrix} 1 & 1 & -1 \\ -2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\right) = (-3) \cdot 8 = -24.$$

(b) Given that $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$, find

$$\det \begin{bmatrix} a+2d & b+2e & c+2f \\ 3g & 3h & 3i \\ d & e & f \end{bmatrix}$$

Answer: Adding a multiple of one row to another leaves the determinant unchanged. Hence

$$\begin{aligned} \det \begin{bmatrix} a+2d & b+2e & c+2f \\ 3g & 3h & 3i \\ d & e & f \end{bmatrix} &= \det \begin{bmatrix} a & b & c \\ 3g & 3h & 3i \\ d & e & f \end{bmatrix} \\ &= 3 \det \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} \\ &= -3 \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= -21. \end{aligned}$$

6. (15%) Suppose that the augmented matrix of a system of linear equations has been partially reduced using elementary row operations to

$$\left[\begin{array}{ccc|c} 1 & 2 & a+2 & b \\ 0 & 1 & b-1 & a \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Find **all values**(if any) of a and b for which the system is inconsistent.

Answer: For an inconsistent system, we must have a row with all but the last entry zero and the last entry nonzero, that is, $a = 0$ and $b \neq 0$

(b) Find **all values** (if any) of a and b for which the system has exactly one solution.

Answer: If $a \neq 0$, Gaussian elimination guarantees a unique solution.

(c) Find **all values** (if any) of a and b for which the system has infinitely many solutions.

Answer: To get a free variable, we must have $a = b = 0$.