Mathematics 1300 Final Exam
April 13, 2009

1. (14) Let $P(-1,1,1)$, $Q(1,2,3)$ and $R(2,1,0)$ be points in R^3.

   (a) Find the equation of the plane in R^3 containing P, Q and R.
   Answer: $Q-P=(2,1,2)$, $R-P=(3,0,-1)$ and $(2,1,2)\times(3,0,-1)=(-1,8,-3)$. Using this as a direction number and evaluating at (any) one point gives$-x+8y-3z=6$. Alternatively, solve $ax+by+cz=d$ for a, b and c. The reduced row echelon form of

   $$A=\begin{bmatrix} -1 & 1 & 1 & d \\ 1 & 2 & 3 & d \\ 2 & 1 & 0 & d \end{bmatrix}$$

   is

   $$A=\begin{bmatrix} 1 & 0 & 0 & -d/6 \\ 0 & 1 & 0 & 4d/3 \\ 0 & 0 & 1 & -d/2 \end{bmatrix}$$

   Hence $-\frac{d}{6}x+\frac{4d}{3}y-\frac{d}{2}=d$, or $x-8y+3z=-6$.

   (b) Find the equation of line L in R^3 passing through P and Q.
   Answer: The line consists of all points of the form:

   $$(-1,1,1)+t(2,1,2) \text{ or, alternatively, } (1,2,3)+t(2,1,2)$$

   (c) Find the area A of the triangle determined by P, Q and R.
   Answer: The area is

   $$\frac{1}{2}\|(-1,8,-3)\|=\frac{1}{2}\sqrt{74}$$

   (d) Find the point of intersection of L and the xy-plane.
   Answer: For the z coordinate of $(-1,1,1)+t(2,1,2)$ to be 0 we need $1+2t=0$, or $t=-\frac{1}{2}$. Then $(-1,1,1)-\frac{1}{2}(2,1,2)=(-2,\frac{1}{2},0)$. Alternatively, for $(1,2,3)+t(2,1,2)$ we get $t=-\frac{3}{2}$ so that $(1,2,3)-\frac{3}{2}(2,1,2)=(-2,\frac{1}{2},0)$.

2. (15) Let

   $$A=\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find the determinant of A.

Answer: Expand on last row and then on the last column to get

$$\det \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1$$

Alternatively, the one elementary row operation $R_3 \leftarrow R_3 - R_1$ makes the matrix upper triangluar. The determinant is then the product of the diagonal elements, all of which are one.

(b) Find A^{-1}.

Answer:

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftarrow R_1 - R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \leftarrow R_2 - R_4$$
$$R_3 \leftarrow R_3 + R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

and so

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Find *all* solutions $\mathbf{x}$ to

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Answer: Multiply the equation by A^{-1} to get

$$\mathbf{x} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(d) Find the adjoint of A.
Answer: Since $\operatorname{adj}(A) = \det(A)\, A^{-1}$, we have $\operatorname{adj}(A) = A^{-1}$.

3. (24) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Evaluate the following matrices:

(a) A^T *Answer*: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

(b) A^{-1} *Answer*: $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

(c) $(A^{-1})^T$ *Answer*: $(A^{-1})^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

(d) $(A^T)^{-1}$ *Answer*: $(A^T)^{-1} = (A^{-1})^T$

(e) $(A^{-1})^2$ *Answer*: $(A^{-1})^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(f) $(A^2)^{-1}$ *Answer*: $(A^2)^{-1} = (A^{-1})^2$

4. (10)

(a) Calculate $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$, the projection of $\mathbf{u}$ along $\mathbf{v}$, where $\mathbf{u} = (2, -2, 3)$ and $\mathbf{v} = (2, 1, 3)$.
Answer:

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{11}{14}(2, 1, 3)$$

(b) Find a unit vector in the direction of $\mathbf{w} = (4, -3, 2, 1)$.

Answer: $\|\mathbf{w}\|^2 = 30$ and so the unit vector in the direction of $\mathbf{w}$ are $\frac{1}{\sqrt{30}}(4, -3, 2, 1)$.

(c) Find all values t so that vectors $(2, 5, -3, 6)$ and $(4, t, 7, 1)$ are orthogonal.

Answer: Orthogonally implies that $0 = (2, 5, -3, 6) \cdot (4, t, 7, 1) = 5t - 7$ and so $t = \frac{7}{5}$.

5. (16) Let $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (0, 1, 1)$.

(a) Find a vector $\mathbf{w}$ in R^3 that is *not* in the span of $\{\mathbf{u}, \mathbf{v}\}$. You *must* justify your answer.

Answer: $c\mathbf{u} + d\mathbf{v} = (c, d, c + d)$. Hence any vector (w_1, w_2, w_3) with $w_3 \neq w_1 + w_2$ is not in the span. For example $\mathbf{w} = (0, 0, 1)$ is not in the span of $\{\mathbf{u}, \mathbf{v}\}$.

(b) Find a vector $\mathbf{w}$ in R^3 so that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for R^3. You *must* justify your answer.

Answer: The set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. Any vector not in the span of $\{\mathbf{u}, \mathbf{v}\}$ will give three linearly independent vectors in R^3, which makes it a basis.

(c) Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be the basis you gave in part (b). Find real numbers a, b and c so that $(25, 13, -17) = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

Answer: We need to solve

$$(25, 13, -17) = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = (a, b, a + b + c)$$

In this case $a = 25$, $b = 13$ and $c = -55$.

(d) Show that $C = \begin{bmatrix} 11 & 17 \\ 1 & 13 \end{bmatrix}$ is in the span of $\{A, B\}$ where $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 3 & 4 \end{bmatrix}$.

Answer: We need to find c and d so that

$$c \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix} + d \begin{bmatrix} 3 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 17 \\ 1 & 13 \end{bmatrix}$$

This translates to four equations in two unknowns:

$$\begin{aligned} 2c + 3d &= 11 \\ -2c + d &= 17 \\ 4c + 3d &= 1 \\ 3c + 4d &= 13 \end{aligned}$$

whose solution is $c = -5$ and $d = 7$.

6. (4) In this question, each answer is a number.

   (a) The smallest possible value for the dimension of the null space of a 17×23 matrix is $\underline{\quad 6 \quad}$.

   (b) The largest possible value of the dimension of the column space of a 17×23 matrix is $\underline{\quad 17 \quad}$.

   (c) Let A be a 17×23 matrix whose null space has dimension 15. The dimension of the row space is $\underline{\quad 8 \quad}$.

   (d) the span of $(1,1,1,1,1)$, $(1,1,0,0,1)$ and $(0,0,1,1,-1)$ has dimension $\underline{\quad 3 \quad}$.

7. (18) The matrix

$$A = \begin{bmatrix} 2 & -10 & 1 & 1 & 22 & -2 & 0 \\ 4 & -20 & 1 & 1 & 36 & 0 & 16 \\ 1 & -5 & 2 & 2 & 23 & 3 & 26 \\ 4 & -20 & -2 & -2 & 12 & 3 & 19 \end{bmatrix}$$

has reduced row echelon form

$$R = \begin{bmatrix} 1 & -5 & 0 & 0 & 7 & 0 & 3 \\ 0 & 0 & 1 & 1 & 8 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find a basis for the null space of A.

   Answer: The columns of A and R correspond to the variables $x_1, \ldots, x_7$. For the free variables, let

$$\begin{aligned} x_2 &= s, \\ x_4 &= t, \\ x_5 &= u, \\ x_7 &= v \end{aligned}$$

   and then

$$\begin{aligned} x_1 &= 5s - 7u - 3v \\ x_3 &= -t - 8u - 4v \\ x_6 &= -5v \end{aligned}$$

   Hence

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5, x_6, x_7) &= s(5,1,0,0,0,0,0) \\ &+ t(0,0,-1,1,0,0,0) \\ &+ u(-7,0,-8,0,1,0,0) \\ &+ v(-3,0,-4,0,0,-5,1) \end{aligned}$$

The basis is then

$$\{(5,1,0,0,0,0,0),(0,0,-1,1,0,0,0),$$
$$(-7,0,-8,0,1,0,0),(-3,0,-4,0,0,-5,1)\}$$

(b) Find a basis for the row space of A. *Answer*: Rows 1,2 and 3 of R.

(c) Find a basis for the column space of A. *Answer*: Columns 1, 3 and 6 of A

(d) The dimension of the null space of A is __4__

(e) The dimension of the row space of A is __3__

(f) The dimension of the column space of A is __3__

8. (12) In each part of this question a *subset* W of the vector space R^5 is defined. State whether or not W is a subspace of R^5. *Justify* your answer.

(a) $W = \{(0, a, 2a+3b, 0, 5a-6b+3c) \mid a, b, c \text{ in } \mathrm{R}\}$. *Answer*: W is closed under addition and scalar multiplication, so it is a subspace of R^5.

(b) $W = \{(0, a, 2a+3, 0, a+b+c) \mid a, b, c \text{ in } \mathrm{R}\}$. *Answer*: The third coordinate of vectors in W is not closed under addition or scalar multiplication, so W is not a subspace. of R^5. Alternatively, since W does not contain the zero vector, W can not be a subspace.

(c) $W = \{(a, b, c, d, e) \mid 7a+5b+3c+2d+e=0\}$. *Answer*: W is closed under addition and scalar multiplication, so it is a subspace of R^5.

9. (7) Indicate whether each statement is true or false:

(a) Whenever a finite set S spans a vector space V, then there is a subset of S which is a basis of V . . . . [X] True [ ] False

(b) If a vector v is not in the span of $\{u_1, u_2, u_3\}$, then the set $\{u_1, u_2, u_3\}$ is linearly independent . . . . [ ] True [X] False

(c) If V and W are subspace of some R^n, then the set $\{v + w \mid v \text{ in } V \text{ and } w \text{ in } W\}$ is a subspace of R^n . . . . [X] True [ ] False

(d) There is a basis for R^6 which contains the following three vectors: $(5,3,4,7,5,6)$, $(0,6,5,4,3,7)$ and $(0,0,9,2,10,3)$ . . . . [X] True [ ] False

(e) $\det(-A) = -\det(A)$ . . . . [ ] True [X] False

(f) $\det(AB) = \det(A)\det(B)$ . . . . [X] True [ ] False

(g) $\det(A+B) = \det(A) + \det(B)$ . . . . ☐ True ☒ False

Suggested point distribution:

1a	4	2a	4	3a	6	4a	4	5a	4	6a	1	7a	4	8a	4	9
1b	4	2b	4	3b	6	4b	3	5b	4	6b	1	7b	4	8b	4	number right −
1c	4	2c	4	3c	6	4c	3	5c	4	6c	1	7c	4	8c	4	number wrong
1d	2	2d	3	3d	6			5d	4	6d	1	7d	2			with minimum
				3e	6							7e	2			0
				3f	6							7e	2			
	14		15		24		10		16		4		18		12	9 total:120