b)Let $a_n = \frac{n^2-1}{2n^2+3}$. Prove **by using the definition** that $\lim_{n\to\infty} a_n = \frac{1}{2}$.

[6]

TAKE $\varepsilon > 0$. WE WANT TO FIND N IN $\mathbb{N}$ SUCH THAT $\forall n \geq N$ WE HAVE $|a_n - \frac{1}{2}| < \varepsilon$. BUT:

$$|a_n - \tfrac{1}{2}| = \left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{2n^2-2-2n^2-3}{2(2n^2+3)}\right| = \left|\frac{-5}{2}\right| \frac{1}{2n^2+3} < \varepsilon \text{ IF}$$

$$2n^2+3 > \frac{2}{5\varepsilon}, \quad n^2 > \frac{1}{2}\left(\frac{2}{5\varepsilon} - 3\right), \quad n > \frac{1}{\sqrt{2}}\sqrt{\frac{2}{5\varepsilon} - 3}$$

(WHENEVER $\frac{2}{5\varepsilon} - 3 > 0$).

$$\text{So, } N = \left[\frac{1}{\sqrt{2}}\sqrt{\frac{2}{5\varepsilon} - 3}\right] + 1 \quad \left(\text{OR } N = 1, \text{ IF } \frac{2}{5\varepsilon} - 3 \leq 0\right)$$

WOULD DO.

2. Find if the following series converge or not. Show all of your work and **state which test** are you using.

a) $\sum_{n=0}^{\infty} (-5) e^{-n^2} n! = (-5) \sum_{n=0}^{\infty} e^{-n^2} n!$. FOR $a_n = e^{-n^2} \cdot n! > 0$, USE

[4]

RATIO TEST: $\frac{a_{n+1}}{a_n} = \frac{e^{-(n+1)^2} (n+1)!}{e^{-n^2} \; n!} = \frac{n+1}{e^{2n+1}} \to 0$, SINCE

$\lim_{x\to\infty} \frac{x+1}{e^{2x+1}} = \left[\frac{\infty}{\infty}\right] \overset{L'Hop}{=} \lim_{x\to\infty} \frac{1}{2e^{2x+1}} = 0$. So $\sum_{n=0}^{\infty} e^{-n^2} n!$ CONVERGES

AND $\sum (-5) e^{-n^2} n!$ MUST ALSO CONVERGE.

b) $\sum_{n=1}^{\infty} \frac{1}{3^n}\left(\ln\left(\frac{n+1}{n}\right)^n\right)^n$, $a_n \geq 0$. ROOT TEST:

[4]

$\sqrt[n]{a_n} = \frac{1}{3} \ln\left(\frac{n+1}{n}\right)^n = \frac{1}{3}\ln\left(1+\frac{1}{n}\right)^n \to \frac{1}{3}\ln e = \frac{1}{3} < 1$

AND SO THE SERIES CONVERGES.

c) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3n(2+\cos n)}$. $a_n \geq 0$. COMPARISON TEST: SINCE $2 + \cos n \leq 3$

[3]

$a_n \geq \frac{\sqrt{n}}{3n \cdot 3} = \frac{1}{9\sqrt{n}} = b_n$, AND $\sum b_n = \frac{1}{9}\sum \frac{1}{\sqrt{n}}$ DIVERGES

(p SERIES, $p = \frac{1}{2}$). THUS $\sum a_n$ DIVERGES ALSO.

3. a)Find the radius of convergence of the power series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^n}{n}$ and show that the series converges conditionally at one end of the interval of convergence and diverges at the other.

[5] $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1$, $R = \frac{1}{1} = 1$, $x_0 = 2$

SERIES CONVERGES ABSOLUTELY ON $(1, 3)$.

$x=3$: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ CONVERGES (ALTERNATING SERIES, $\lim_{n\to\infty}\frac{1}{n}=0$, AND $\left\{\frac{1}{n}\right\}$ DECREASING) BUT DOES NOT CONVERGE ABSOLUTELY: $\sum\frac{1}{n}$ DIVERGES

$x=1$: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} = \sum(-1)^{2n+1}\frac{1}{n} = -\sum\frac{1}{n}$ DIVERGES ($p=1$, P SERIES)

b) Let $\sum a_n(x-x_0)^n$ be a power series with a positive (finite) radius of convergence R. Show that if the series converges absolutely at one end of the interval of convergence, then it must also converge absolutely at the other end of the interval of convergence.

[5]

THE END POINTS OF THE INTERVAL ARE: X_0-R AND X_0+R.

IF THE SERIES CONVERGES ABSOLUTELY AT X_0-R, THEN

$\sum|a_n(x_0-R-x_0)^n| = \sum|a_n(-R)^n| = \sum|a_n|R^n$ CONVERGES.

BUT THEN AT X_0+R: $\sum|a_n(x_0+R-x_0)^n| = \sum|a_n|R^n$

AND THE SERIES IS THE SAME AS THE ONE ABOVE AND THUS CONVERGES.

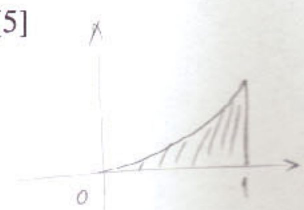
(THE PROOF IS SIMILAR IF WE ASSUME ABSOL. CONV. AT X_0+R FIRST.)

4. a) State the part of the Theorem on Integration of Power Series that refers to definite integrals.

b) Show that the area under the graph of the function $x^2\sin x$ and between x=0 and x=1 equals to

$$\frac{1}{4}-\frac{1}{3!6}+\frac{1}{5!8}-\ldots+\frac{(-1)^n}{(2n+1)!(2n+4)}+\ldots\ .$$

[5]



$$A=\int_0^1 x^2\sin x\,dx \quad ; \quad \sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$x^2\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+3}}{(2n+1)!} \quad \text{By a):}$$

$$A=\int_0^1 x^2\sin x\,dx=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\int_0^1 x^{2n+3}dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left.\frac{x^{2n+4}}{2n+4}\right|_0^1=$$

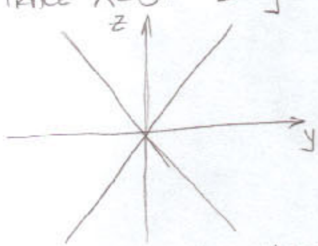
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{2n+4}-0=\frac{1}{4}-\frac{1}{3!}\cdot\frac{1}{6}+\frac{1}{5!}\cdot\frac{1}{8}-\ldots$$

5. Let $\mathbf{r}(t) = \begin{cases} \left(e^{-t}, \dfrac{\sin 2t}{t}, \sqrt{2e^{-2t} + \dfrac{\sin^2 2t}{t^2}}\right) & ,t \neq 0 \\ (1,0,\sqrt{2}) & ,t = 0 \end{cases}$, and let S be a quadric surface given by $z^2 = 2x^2 + y^2$.

a) Classify the surface and draw a rough sketch of it by first drawing two dimensional pictures of the traces in the planes x=0 and z=2 (or z=-2).

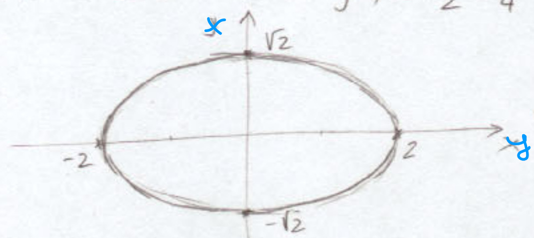
[6]

TRACE $X=0$: $z^2 = y^2$

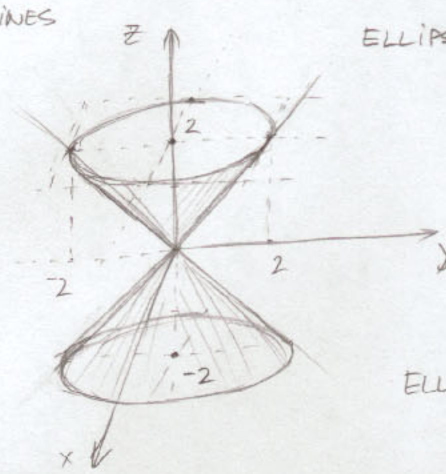


TWO INTERSECTING LINES

TRACE $z=2$: $4 = 2x^2 + y^2$, $1 = \frac{x^2}{2} + \frac{y^2}{4}$



ELLIPSE



ELLIPTIC CONE

b) Show that the graph of the function $\mathbf{r}(t)$ is a curve that lies on the surface S and find $\lim_{t \to 0} \mathbf{r}(t)$.

[5]

$t \neq 0$: $z^2 = 2e^{-2t} + \frac{\sin^2 2t}{t^2} = 2x^2 + y^2$, $t=0$: $(\sqrt{2})^2 = 2 = 2 \cdot 1 + 0$

$$\lim_{t \to 0} \underline{r}(t) = \left\langle \lim_{t \to 0} e^{-t}, \lim_{t \to 0} \frac{\sin 2t}{t}, \lim_{t \to 0} \sqrt{2e^{-2t} + \frac{\sin^2 2t}{t^2}} \right\rangle =$$

$$= \left\langle e^{-0}, 2 \lim_{t \to 0} \frac{\sin 2t}{2t}, \sqrt{2 e^{0} + 4 \lim_{t \to 0} \left(\frac{\sin 2t}{2t}\right)^2} \right\rangle = \langle 1, 2, \sqrt{6} \rangle$$

c) Is $\mathbf{r}(t)$ continuous or differentiable at t=0? Explain.

[3]

No: $\lim_{t \to 0} \underline{r}(t) = \langle 1, 2, \sqrt{6} \rangle \neq \langle 1, 0, \sqrt{2} \rangle = \underline{r}(0)$

AND SO $\underline{r}(t)$ IS NOT CONTINUOUS AT $t=0$.

$\underline{r}(t)$ IS NOT DIFFERENTIABLE AT $t=0$, SINCE IT IS NOT CONTINUOUS AT $t=0$.