136.275, Assignment No. 1

September 22, 2004

The assignment is due Wednesday, September 29, 2004 in class. Late assignments receive a mark zero. **Show all of your work**.

1. Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} a_n = L$ and let $L \neq 0$.
   a) Prove that eventually $a_n \neq 0$, i.e. that there exists an N in $\mathbb{N}$ such that for every $n \geq N$ we have that $a_n \neq 0$. (Hint: Show that, eventually, $|a_n| > \frac{|L|}{2}$.) [5]

   b) Prove by using the definition of the limit that $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$. [5]

2. a) Prove by using the definition of the limit that if $0 \leq r < 1$, we have that

   $$\lim_{n\to\infty} r^n = 0. \quad [4]$$

   b) State the theorem by which part a) implies that $\lim_{n\to\infty} r^n = 0$ for $-1 < r \leq 0$. [2]

   c) Give an example of a sequence to show that the convergence of $\{ |a_n| \}$ does not imply the convergence of $\{ a_n \}$. [2]

3. Determine if the sequence $\{a_n\}$ converges or not, and if it does, find the limit:
   a) $a_n = \sqrt{n^2+1} - \sqrt{n^2+5}$, [3]
   b) $a_n = \dfrac{(\ln n)^2}{n}$, [3]
   c) $a_n = 3e^n + \sin(n^2+1)$, [3]
   d) $a_n = \dfrac{1^2}{n^3} + \dfrac{2^2}{n^3} + \ldots + \dfrac{n^2}{n^3}$. [3]

4. a) Use the Squeeze Theorem to show the convergence of the sequence $a_n = \dfrac{2^n}{n!}$. [4]

   b) Consider the sequence $\{a_n\}$ whose n-th term is $a_n = \dfrac{1}{n}\sum_{k=1}^{n} \dfrac{1}{1+(k/n)}$. Show that $\lim_{n\to\infty} a_n = \ln 2$ by interpreting a_n as the Riemann sum of a definite integral. State all of the theorems on integrals that you are using. [5]

5. Give the definition of $\lim_{n\to\infty} a_n = \infty$ and use it to prove that $\lim_{n\to\infty} 2\ln(5n^2-3) = \infty$. [5]

Total [44/42]