MATHEMATICS 136.275
EXAM
3 hours.
April 17, 2003

Instructions: Attempt all questions. The total value of all questions is 100. Values of individual questions are printed beside the statement of the question. If you need more space use the reverse side of the page, but indicate clearly that you are doing so. There are two blank pages at the end of the test for you to use as scrap paper.
Please fill in the information requested below. Good luck!

Name ____

Signature _____________________________________

Student No. ___________________________________

QUESTION#	MARK
1	/8
2	/12
3	/9
4	/12
5	/10
6	/11
7	/8
8	/10
9	/9
10	/11
TOTAL	/100

Value

[2] 1. a) Write the definition of the limit of a sequence.

[6] b) Let $\lim_{n\to\infty} a_n = L$. Prove **by using the definition** of the limit that then $\lim_{n\to\infty} a_n^2 = L^2$.
(Hint: use the fact that since $\{a_n\}$ converges, it must be bounded, i.e. $\exists M > 0$ such that $|a_n| \le M$ for all n in $\mathbb{N}$.)

2. Find if the following series converge or not. Show all of your work and **state which test** are you using.

[4] a) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

[4] b) $\sum_{n=1}^{\infty} \frac{\pi n^2}{1+n\sqrt{n}}$

c) $\sum_{n=1}^{\infty} \frac{\cos n}{n^{1+\sqrt{n}}}$

[4]

3. a) State the alternating series test.

[2]

b) Find the radius and the interval of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{\ln n}}\left(\frac{x-3}{2}\right)^n.$$

[7]

4. **Using the series expansion** of f(x)= ln (1+x) about 0, answer the following:

a) What does $f^{(17)}$ (o) equal to? Explain.

[2]

b) Does $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\frac{1}{2^n}$ converge absolutely, conditionally or it diverges?

[2] Explain.

c) What is the series expansion of g(x) = x ln (1+x) about 0?

[2]

d) Using the derivative of g , explain why is it that

?

[4]

e) Can g be expanded into a power series about x= -1? Explain.

[2]

5. a) Write a formula for the curvature k(t) of the $\mathbb{R}^3$ curve **r**(t).

[2]

b) Let **r**(t) =(cost , $\frac{1}{\sqrt{5}}$sint, $\frac{2}{\sqrt{5}}$sint). Show that the curve has constant curvature 1.

[5]

c) Show that the curve in b) is on the surfaces z=2y and $x^2 + y^2 + z^2 = 1$.
Guess what kind of a curve it is.

[3]

6. Let $f(x,y) = \sqrt[3]{xy}$.

a) Is f(x,y) continuous at (0,0) ? Show your work.

[2]

b) Show that the directional derivative $D_{\mathbf{u}}$ of f at (0,0) exists only if the unit vector **u** is parallel to either x or y axis. (Use the definition of directional derivative.)

[5]

c) Show that f is not differentiable at (0,0) by using the definition of the derivative. (You have f_x and f_y at (0,0) from b).)

[4]

7. Let f(x,y) = $3xe^{y} - x^{3} - e^{3y}$.

a) Show that f has only one critical point and that f has a local maximum there by using the Second Partial Derivatives test.

[6]

b) Write the equation of the tangent plane to the graph of f at the point (1, 0, 1).

[2]

8. Let R be the square bounded by the lines x+y=1, x+y=-1, x-y=1 and x-y= -1. Let the transformation T^{-1} from (x,y) plane to the (u,v) plane be given by u=x+y and v=x-y .

[3] a) Draw R in the xy plane and S in the uv plane, where S= T^{-1}(R) .

[3] b) Find the transformation T , i.e. find x=x(u,v) and y=y(u,v), and show that the Jacobian of T equals to $-\frac{1}{2}$.

[4] c) Show that for any continuous function f , $\iint_R f(x+y)dA = \int_{-1}^{1} f(u)du$ by using the change of variable theorem and T from above.

9. Let G be the solid inside the sphere $x^2 + y^2 + z^2 = 2$,
outside the cone $z = \sqrt{x^2 + y^2}$ and above the plane z=0. Express the volume of G **without calculating it** as a triple iterated integral:

a) In spherical coordinates,

[4]

b) In cylindrical coordinates..

[5]

10. Let **F** (x,y) = $\left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$.

a) Prove that **F** is a conservative field on any closed disk that does not include the origin, by using a part of a theorem from class.

[4]

b) If the curve C_1 is the curve given by $(x-2)^2+(y-2)^2=1$, calculate the work done by **F** on a particle that moves along C_1 in a counterclockwise direction. (Use a) and a part of a theorem from class.)

[2]

c) Show that the work done by **F** on a particle that moves counterclockwise along the curve C_2 given by x= a cost, y= a sint, $0 \leq t \leq 2\pi$, and a>0, equals to 2π. Does this contradict a) ? Explain.

[5]